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On Application of Absorbing Markov Chain Model to Determine Students' Academic Progress: A Case Study of Mathematics Department, Ekiti State University, Nigeria

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Abstract. This paper investigates students' enrolment pattern and their academic performance as they progress from the point of admission to the point of graduation focusing on the Mathematics Department, Ekiti State University, Ado Ekiti in Nigeria. In this research work the Transition Probability Matrix (TPM) was formulated over sequent periods of academic calendars and the probabilities of absorption that include both the Graduating Students and Withdrawn Students were captured. The paper is geared towards the obtainment of the fundamental matrix that will be employed to determine the anticipated length of stay of students in the institution before graduation. The analysis of the requisites was executed by employing the Markov chain model to student populations at different levels in the University including the freshmen, the sophomore transfers, the junior transfers, and the finalists. This was carried out to estimate the differences in the cohorts' behavior over the entire academic period. Based on the results and findings after the data smoothening, future predictions were made on the enrolment into the department as well as the academic performance of the students. Hinging on the analysis includes some prescriptions such as an intentional increase in the rate of retention of the freshmen and the boost given to the sophomores to take on the area with comparative advantage rather than leaving the institution. Other post-analysis discussions and necessary recommendations are included without overlooking the drawbacks of the results from the analysis.

Keywords: Transition Probability Matrix, Markov Chain, Fundamental Matrix, Probability of Absorption

Introduction

The students' academic performance and progress evaluation is an essential facet of all educational system. It depicts a clear understanding of how the students resonate and it can be used as supporting parameters for important decisions and planning. This can be quite helpful for the management of the institution to put into place educational policies that will ensure optimum positioning of the institution in the global academic market. Results from prior similar findings suggest that the student's progress through their education toward acquiring degrees possesses apposite stochastic features that can be modeled primly as a Markov chain.

Student's academic performance and status are measured by their academic strength which is rated based on the Cumulative Grade Point Average (CGPA) that are calculated at the end of every semester. For instance, if a student in the 100 level, 200 level, or 300 level has a $CGPA < 1.0$ at the end of the second semester of a particular level, the university academic board assumes that such a student is weak academically for that course of study and might not be able to continue with such a program hence, the candidate is placed on probation.

The probation period is to alert such a student that if there is no improvement academically in the next semester or session the studentship of such a student will automatically be withdrawn by the system basically on academic grounds. For students with $CPGA > 1.5$, the academic strength performance is measured by the CGPA is out of the total obtainable CGPA of 5 points and this determines the class of degree to which the student finishes with.

In the theory of probability, the rationale behind the stochastic or random process is the mathematical object that is basically the collection of the defined random variables. The random variables are indexed or associated with a set of numbers that are usually dimmed at points in time. More so, the Markov process can be termed as a stochastic process or random process that is indexed by time and has the tendency that given the present, the future is independent of the past. Hence, a discrete-time Markov chain can be described as a stochastic process that is the generalized simplest form of an independent sequence of random variables.

According to Tijms (2003), the Markov Chain is a random sequence in which the dependency of successive events can only go back by one unit in time. The application and adoption of Markov Chain has cut through edges in different fields of Natural and Physical Sciences, Medical Sciences, and Engineering. It has become relevant for the development of models to survey and analyze the progress and performances of students in institutions of higher learning.

Applying Markov chain to analyze students' progress and performances provide a means for predicting the population of students that will either graduate or drop out owing to age, gender, or broad field of study. The model tends to estimate the length of time a student will remain in the academic system, measures the probability of a student completing a course of study in due time, and the average time a student will take to finish the program.

Spread across the last two decades are lots of studies carried out in this area that have proven to be beneficial to mankind. Such studies include the application of the Markov chain in analyzing to predict the gaps filled by the mathematical achievement that exist between the African Americans and white American students as opined by Moody and DuClouy (2014). Hlavatý and Dömeová (2014) have introduced the Markov chain model to students' academic progress in pursuant of their course of studies. The Non-Homogeneous Markov Systems (NHMS) theory with fuzzy states for reporting the students' educational performance in Greek Universities was the focus of Symeonaki and Kalamatianou (2011). On the other hand, Gani (1963) employed the input out-put models application to propose and forecast the enrollment pattern for awarding bachelor's degrees in Australian Universities and Poland and Scheraga (1970) adopted a variant of the Markov chain model to understudy higher education in Australia. The performance and students' progression during their study in the higher institutions were modeled by various authors as Adam (2015), Oke, M. O., *et al*., (2020), Adeleke, Oguntuase, and Ogunsakin (2014), Al-Awadhi and Ahmed (2002), Konsowa and Al-Awadhi (2010), Al-Awadhi and Konsowa (2007), Auwalu, Mohammed, and Saliu (2013), Mashat, Ragab, and Khedra (2012), Shah and Burke (1999), using an absorbing Markov chain.

Several authors utilized the input-output application models to project enrolment for bachelor's degree awards in Australian higher institutes of learning. While in years back, another version of the input-output application model was adopted to project the progress of higher institutes of learning in Australia. Markov Chain can be used to predict what the enrolments for an educational system will be. Similar Markov chain models used for predictions were adopted in the New South Wales State Government educational system for over two decades and the envisaged enrolments were compared to the actual enrolments in the years in view.

According to Shah and Burke (1999), the need and the use of educational indicators were analyzed concerning what is obtainable in Swaziland. Adebanji, *et al*. (2021), examined the trend of students' performances considering the psychomotor domain. The data requisites are carefully weighed alongside the implications of the policies that establish the fixed series and the construction of an input-output academic table that can deduce a set of indicators that will be generally embraced and championed a Markov chain model developed to forecast what will beget the Swaziland educational system compared with such that is aimed at describing the Swaziland system of education.

Methodology

Theorem 1: Any finite Markov Chain that is independent of the starting state will tends to ∞ as the probability after n steps approaches 1 are in an ergodic state.

Corollary: In an absorbing Markov chain, the probability of absorption is 1.

Proof: Let D be an underlying digraph of the transition digraph T. Each of the transient vertices u_i can reach an ergodic vertex u_j because the ergodic sets form a contrabasis vertex for the condensation D^*.

Since the transition digraph has finite vertices, there is a number r such that for each transient vertices u_i, there is some ergodic vertices u_j where $d(u_i, u_j) \leq r$, with $d(u_i, u_j)$ has the shortest path linking u_i to u_j in D. Thus, the probability of an absorbing Markov chain entering an ergodic vertex that is independent of the starting state at most in r steps has the probability P.

Therefore, for a Markovian process, the probability of the absorbing Markov chain not getting to the ergodic vertex state in r steps is $1 - P$. While the probability of the absorbing Markov chain not entering an ergodic vertex in kr time steps is $(1 - P)^k$. As stated by Hlavatý and Dömeová (2014), since $0 < P \leq 1$, the probability of the Markov chain not reaching an ergodic state tends to 0 as $k \to \infty$.

Before presenting other supporting theorems, there is a need to present a canonical form for the absorbing transition matrices that is achieved by partitioning the transition matrix into recognizable blocks and as usual, the absorbing states will be the one to come last in the block.

The absorbing Markov chain theory is the model for which analyzing the students' academic progress and performance hinges on. The general canonical form of the Probability Transition Matrix (PTM) of the absorbing Markov chain with n transient states and r absorbing states is given as:

$$P = \begin{pmatrix} Q & \vdots & R \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I \end{pmatrix}_{(n+r)\times(n+r)} \quad (1)$$

where Q is an $n \times n$ matrix that represents the transition probability from one non-absorbing state to another non-absorbing state. The summation of the entries along each of the row is less than one. Matrix R is an $n \times r$ matrix. It contains the transiting probabilities that goes from a non-absorbing state to an absorbing state. The matrix 0 is an $r \times n$ zero matrix that depicting a transition from an absorbing state to a non-absorbing state which is rather impossible. Lastly, I is an $r \times r$ identity matrix with $P_{ii} = 1$ and all other entries $P_{ij} = 0$. This depicts that once the transition matrix enters the absorbing state, it is not possible to transit the system to another state let alone entering into another absorbing state. Hence, the transition matrix is in the canonical form.

Fundamental Matrix Computation and its Requisites

For an absorbing Markov chain P, the matrix $M = (1 - Q)^{-1}$ is tagged as the fundamental matrix for P. The entries, n_{ij}, of the matrix M determines the expected times that the entire process will be in the transient state, j, if it has commenced from the transient state i as:

$$M = (I - Q)^{-1} \quad (2)$$

where $I = n \times n$ is an identity matrix with a different order from the identity matrix mentioned (1) that is of order $r \times r$.

Before expressing the dynamic that could be used to compute the entries of the fundamental matrix, $M = (I - Q)^{-1}$ it is expedient to consider the theorem associated with it and its corresponding proof where M is the fundamental matrix, I is the identity matrix, Q is a Matrix.

Theorem 2:

If $P = \begin{pmatrix} Q & \vdots & R \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I \end{pmatrix}$, then the Fundamental Matrix $M = (m_{ij})$ is will be given as:

$$m_{ij} = \begin{cases} \frac{1}{1-P_{ij}}, & if\ i = j \\ \frac{\prod_{r=1}^{j-1} P_{r(r+1)}}{\prod_{r=1}^{j}(1-P_{rr})}, & if\ i < j \\ 0, & if\ i > j \end{cases} \tag{3}$$

Proof

The proof of this theorem shall be established using the principle of mathematical induction.

Let the $n \times n$ matrices Q and M be denoted by Q_n and M_n respectively.

For $n = 2$, Q_n have

$$Q_2 = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} \tag{4}$$

$$(I - Q_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} = \begin{pmatrix} 1 - P_{11} & -P_{12} \\ 0 & 1 - P_{22} \end{pmatrix} \tag{5}$$

Let $a_{ii} = 1 - P_{ii}$ and $b_{i(i+1)} = -P_{i(i+1)}$ (6)

$$(I - Q_2) = \begin{pmatrix} a_{11} & b_{12} \\ 0 & a_{22} \end{pmatrix} \tag{7}$$

On evaluating the determinant of (5) and comparing with (7), one obtains

$$|I - Q_2| = (1 - P_{11})(1 - P_{22}) = a_{11} \times a_{22} \tag{7*}$$

The inverse of $(I - Q_2)$ is then given as

$$(I - Q_2)^{-1} = M_2 = \frac{1}{a_{11} \times a_{22}} \begin{pmatrix} a_{22} & -b_{12} \\ 0 & a_{11} \end{pmatrix} = \begin{pmatrix} a_{22} \times \frac{1}{a_{11} \times a_{22}} & -b_{12} \times \frac{1}{a_{11} \times a_{22}} \\ 0 \times \frac{1}{a_{11} \times a_{22}} & a_{11} \times \frac{1}{a_{11} \times a_{22}} \end{pmatrix} \tag{8}$$

On evaluating and substituting (6) in (8) gives

$$M_2 = \begin{pmatrix} \frac{1}{a_{11}} & \frac{-b_{12}}{a_{11} \times a_{22}} \\ 0 & \frac{1}{a_{22}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-P_{11}} & \frac{P_{12}}{(1-P_{11})(1-P_{12})} \\ 0 & \frac{1}{1-P_{22}} \end{pmatrix} \tag{9}$$

Since it is valid for $n = 2$, let it be tried for $n = 3$. Then, we obtain

$$Q_3 = \begin{pmatrix} P_{11} & P_{12} & 0 \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{pmatrix} \tag{10}$$

$$(I - Q_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} & 0 \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{pmatrix} = \begin{pmatrix} 1 - P_{11} & -P_{12} & 0 \\ 0 & 1 - P_{22} & -P_{23} \\ 0 & 0 & 1 - P_{33} \end{pmatrix} \tag{11}$$

Substituting (6) in (11) gives rise to

$$(I - Q_3) = \begin{pmatrix} a_{11} & b_{12} & 0 \\ 0 & a_{22} & b_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \tag{12}$$

Evaluating the determinant of (12) gives

$$|I - Q_3| = (1 - P_{11})(1 - P_{22})(1 - P_{33}) = a_{11} \times a_{22} \times a_{33} \tag{13}$$

$$\text{Cofactor of } (I - Q_3) = \begin{pmatrix} a_{22}a_{23} & 0 & 0 \\ -b_{12}a_{33} & a_{11}a_{33} & 0 \\ b_{12}b_{23} & -a_{11}b_{23} & a_{11}a_{22} \end{pmatrix} \tag{14}$$

$$(I - Q_3)^T = \begin{pmatrix} a_{22}a_{23} & -b_{12}a_{33} & b_{12}b_{23} \\ 0 & a_{11}a_{33} & -a_{11}b_{23} \\ 0 & 0 & a_{11}a_{22} \end{pmatrix} \tag{15}$$

$$(I - Q_3\,)^{-1} = M_3 = \frac{1}{a_{11}a_{22}a_{33}} \begin{pmatrix} a_{22}a_{23} & -b_{12}a_{33} & b_{12}b_{23} \\ 0 & a_{11}a_{33} & -a_{11}b_{23} \\ 0 & 0 & a_{11}a_{22} \end{pmatrix}$$

$$M_3 = (I - Q_3\,)^{-1} = \begin{pmatrix} a_{22}a_{23}\dfrac{1}{a_{11}a_{22}a_{33}} & -b_{12}a_{33}\dfrac{1}{a_{11}a_{22}a_{33}} & b_{12}b_{23}\dfrac{1}{a_{11}a_{22}a_{33}} \\ 0 & a_{11}a_{33}\dfrac{1}{a_{11}a_{22}a_{33}} & -a_{11}b_{23}\dfrac{1}{a_{11}a_{22}a_{33}} \\ 0 & 0 & a_{11}a_{22}\dfrac{1}{a_{11}a_{22}a_{33}} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} \frac{a_{22}a_{23}}{a_{11}a_{22}a_{33}} & \frac{-b_{12}a_{33}}{a_{11}a_{22}a_{33}} & \frac{b_{12}b_{23}}{a_{11}a_{22}a_{33}} \\ 0 & \frac{a_{11}a_{33}}{a_{11}a_{22}a_{33}} & \frac{-a_{11}b_{23}}{a_{11}a_{22}a_{33}} \\ 0 & 0 & \frac{a_{11}a_{22}}{a_{11}a_{22}a_{33}} \end{pmatrix} = \begin{pmatrix} \frac{1}{a_{11}} & \frac{-b_{12}}{a_{11}a_{22}} & \frac{b_{12}b_{23}}{a_{11}a_{22}a_{33}} \\ 0 & \frac{1}{a_{22}} & \frac{-b_{23}}{a_{22}a_{33}} \\ 0 & 0 & \frac{1}{a_{33}} \end{pmatrix} \tag{16}$$

Then, the entries of the fundamental 3×3 matrix is computed below:

$$M_3 = \begin{pmatrix} \frac{1}{a_{11}} & \frac{-b_{12}}{a_{11}a_{22}} & \frac{b_{12}b_{23}}{a_{11}a_{22}a_{33}} \\ 0 & \frac{1}{a_{22}} & \frac{-b_{23}}{a_{22}a_{33}} \\ 0 & 0 & \frac{1}{a_{33}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1-P_{11}} & \frac{P_{12}}{(1-P_{11})(1-P_{22})} & \frac{P_{12}P_{23}}{(1-P_{11})(1-P_{22})(1-P_{33})} \\ 0 & \frac{1}{1-P_{22}} & \frac{P_{23}}{(1-P_{22})(1-P_{33})} \\ 0 & 0 & \frac{1}{1-P_{33}} \end{pmatrix} \tag{17}$$

Since the statement holds for $n = 3$, let it be verified for $n = 4$. Then, one obtains

$$Q_4 = \begin{pmatrix} P_{11} & P_{12} & P_{13} & 0 \\ 0 & P_{22} & P_{23} & P_{24} \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & 0 & P_{44} \end{pmatrix} \tag{18}$$

$$(I - Q_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} & P_{13} & 0 \\ 0 & P_{22} & P_{23} & P_{24} \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & 0 & P_{44} \end{pmatrix} \tag{19}$$

$$(I - Q_4) = \begin{pmatrix} 1 - P_{11} & -P_{12} & -P_{13} & 0 \\ 0 & 1 - P_{22} & -P_{23} & -P_{24} \\ 0 & 0 & 1 - P_{33} & -P_{34} \\ 0 & 0 & 0 & 1 - P_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & b_{12} & b_{13} & 0 \\ 0 & a_{22} & b_{23} & b_{24} \\ 0 & 0 & a_{33} & b_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \tag{20}$$

$$|I - Q_4| = (1 - P_{11})(1 - P_{22})(1 - P_{33})(1 - P_{44}) = a_{11} \times a_{22} \times a_{33} \times a_{44} \tag{21}$$

Cofactor of $(I - Q_4) =$

$$\begin{pmatrix} a_{22}a_{33}a_{44} & 0 & 0 & 0 \\ -b_{12}a_{33}a_{44} & a_{11}a_{33}a_{44} & 0 & 0 \\ b_{12}b_{23}a_{44} & -a_{11}b_{23}a_{44} & a_{11}a_{22}a_{44} & 0 \\ -b_{12}b_{23}b_{34} & b_{12}(b_{23}b_{34} - a_{33}b_{24}) - b_{13}a_{22}b_{34} & -a_{11}b_{34}a_{22} & a_{11}a_{22}a_{33} \end{pmatrix} \tag{22}$$

$$(I - Q_4)^T =$$

$$\begin{pmatrix} a_{22}a_{33}a_{44} & -b_{12}a_{33}a_{44} & b_{12}b_{23}a_{44} & -b_{12}b_{23}b_{34} \\ 0 & a_{11}a_{33}a_{44} & -a_{11}b_{23}a_{44} & b_{12}(b_{23}b_{34}-a_{33}b_{24})-b_{13}a_{22}b_{34} \\ 0 & 0 & a_{11}a_{22}a_{44} & -a_{11}b_{34}a_{22} \\ 0 & 0 & 0 & a_{11}a_{22}a_{33} \end{pmatrix} \quad (23)$$

$$(I-Q_4)^{-1} = \frac{(I-Q_4)^T}{|I-Q_4|} \quad (24)$$

$$(I-Q_4)^{-1} = \frac{1}{a_{11}a_{22}a_{33}a_{44}} \begin{pmatrix} a_{22}a_{33}a_{44} & -b_{12}a_{33}a_{44} & b_{12}b_{23}a_{44} & -b_{12}b_{23}b_{34} \\ 0 & a_{11}a_{33}a_{44} & -a_{11}b_{23}a_{44} & b_{12}(b_{23}b_{34}-a_{33}b_{24})-b_{13}a_{22}b_{34} \\ 0 & 0 & a_{11}a_{22}a_{44} & -a_{11}b_{34}a_{22} \\ 0 & 0 & 0 & a_{11}a_{22}a_{33} \end{pmatrix}$$

$$(I-Q_4)^{-1} = M_4 = \begin{pmatrix} \frac{1}{a_{11}} & \frac{-b_{12}}{a_{11}a_{22}} & \frac{b_{12}b_{23}}{a_{11}a_{22}a_{33}} & \frac{-b_{12}b_{23}b_{34}}{a_{11}a_{22}a_{33}a_{44}} \\ 0 & \frac{1}{a_{22}} & \frac{-b_{23}}{a_{22}a_{23}} & \frac{b_{12}(b_{23}b_{34}-a_{33}b_{24})}{a_{11}a_{22}a_{33}a_{44}} - \frac{b_{13}b_{34}}{a_{11}a_{33}a_{44}} \\ 0 & 0 & \frac{1}{a_{33}} & \frac{-b_{34}}{a_{33}a_{44}} \\ 0 & 0 & 0 & \frac{1}{a_{44}} \end{pmatrix} \quad (25)$$

Then, the fundamental 4×4 matrix entries can be calculated below:

$$M_4 = \begin{pmatrix} \frac{1}{1-P_{11}} & \frac{P_{12}}{(1-P_{11})(1-P_{22})} & \frac{P_{12}P_{23}}{(1-P_{11})(1-P_{22})(1-P_{33})} & \frac{P_{12}P_{23}P_{34}}{(1-P_{11})(1-P_{22})(1-P_{33})(1-P_{44})} \\ 0 & \frac{1}{1-P_{22}} & \frac{P_{23}}{(1-P_{22})(1-P_{33})} & \frac{P_{23}P_{34}}{(1-P_{22})(1-P_{33})(1-P_{44})} \\ 0 & 0 & \frac{1}{1-P_{33}} & \frac{P_{34}}{(1-P_{33})(1-P_{44})} \\ 0 & 0 & 0 & \frac{1}{1-P_{44}} \end{pmatrix} \quad (26)$$

Since the assertion is true for $n = 4$, it is assumed that it will be true for $n = k$. Then, it will be shown to be true for $n = k + 1$.

$$Q_{k+1} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & 0 & 0 \\ 0 & P_{22} & P_{23} & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & P_{kk} & P_{kk+1} \\ 0 & 0 & 0 & & 0 & P_{(k+1)(k+1)} \end{pmatrix} \quad (27)$$

$$(I-Q_{k+1}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & 0 & 0 \\ 0 & P_{22} & P_{23} & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & P_{kk} & P_{kk+1} \\ 0 & 0 & 0 & & 0 & P_{(k+1)(k+1)} \end{pmatrix}$$

$$(I-Q_{k+1}) = \begin{pmatrix} 1-P_{11} & -P_{12} & -P_{13} & \cdots & 0 & 0 \\ 0 & 1-P_{22} & -P_{23} & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1-P_{kk} & -P_{kk+1} \\ 0 & 0 & 0 & & 0 & 1-P_{(k+1)(k+1)} \end{pmatrix} \quad (28)$$

$$(I-Q_{k+1}) = \begin{pmatrix} a_{11} & b_{12} & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & b_{23} & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{kk} & b_{k(k+1)} \\ 0 & 0 & 0 & & 0 & a_{(k+1)(k+1)} \end{pmatrix} \quad (29)$$

$$|I-Q_{k+1}| = a_{11}a_{22}a_{33}\ldots a_{kk}a_{(k+1)(k+1)} = \prod_{r=1}^{k+1} a_{rr} \quad (30)$$

Then, transposing (30) gives:

$$(I-Q_{k+1})^T = \begin{cases} \prod_{r=1}^{k+1} a_{rr} & if\ i=j \\ \prod_{r=1}^{j-1}(-b_{r(r+1)} \prod_{r=1}^{k+1} a_{rr}), & if\ i<j \\ 0, & if\ i>j \end{cases} \tag{31}$$

$$M_{k+1} = \frac{(I-Q_{k+1})^T}{|I-Q_{k+1}|} = \frac{1}{\prod_{r=1}^{k+1} a_{rr}} \times \begin{cases} \prod_{r=1}^{k+1} a_{rr} & if\ i=j \\ \prod_{r=1}^{j-1}(-b_{r(r+1)} \prod_{r=1}^{k+1} a_{rr}), & if\ i<j \\ 0, & if\ i>j \end{cases} \tag{32}$$

$$M_{k+1} = \begin{cases} \frac{1}{a_{ij}}, \\ \frac{\prod_{r=1}^{j-1} -b_{r(r+1)}}{\prod_{r=1}^{j} a_{rr}}, \\ 0, \end{cases} = \begin{cases} \frac{1}{1-P_{ij}}, & if\ i=j \\ \frac{\prod_{r=1}^{j-1} P_{r(r+1)}}{\prod_{r=1}^{j}(1-P_{rr})}, & if\ i<j \\ 0, & if\ i>j \end{cases} \tag{33}$$

To ease the computational process, the entries of (33) can be put in the form:

$$M_{k+1} = \begin{cases} \frac{1}{1-P_{ij}}, & if\ i=j \\ a_j m_{i(j-1)}, & if\ i<j \\ 0, & if\ i>j \end{cases} \tag{34}$$

where $a_1 = \frac{P_{1-ij}}{1-P_{11}}$.

Comprehensively, the entries of M are given as

$$\left.\begin{aligned} m_{11} &= \frac{1}{1-P_{11}} \\ m_{12} &= \frac{P_{12}}{(1-P_{11})(1-P_{22})} = a_2 m_{11} \\ m_{13} &= \frac{P_{12}P_{23}}{(1-P_{11})(1-P_{22})(1-P_{33})} = a_3 m_{12} \\ m_{14} &= \frac{P_{12}P_{23}P_{34}}{(1-P_{11})(1-P_{22})(1-P_{33})(1-P_{44})} = a_4 m_{13} \\ &\vdots \\ m_{1k} &= \frac{P_{12}P_{23}P_{34}\dots P_{(k-1)k}}{(1-P_{11})(1-P_{22})(1-P_{33})\dots(1-P_{kk})} = a_k m_{1(k-1)} \end{aligned}\right\} \tag{35}$$

Thus, the same procedure applies to $2, 3, \dots, k-1$ and the k rows of the Fundamental Matrix M. Higher orders of the Fundamental Matrix could be gotten by using various relevant software.

Time of Absorption

The time of absorption is to be determined to be sure of when the Markov chain is absorbed fully into the system. Let μ_i stand for the number of steps to be taken before the Markov chain can be absorbed into any of the absorbing states after the absorption from the transient state, i, has commenced, and let μ be the column vector whose ith entry is μ_i then

$$\mu = MC \tag{36}$$

where M stands for the fundamental matrix and C stands for the column vector with all its entries being $1's$.

Absorption Probability

Let B represents the probability that an absorbing Markov chain will be absorbed into the absorbing state j provided the absorption process begins from the transient state i, and the entries of the matrix be represented by b_{ij}, then B is an $n \times n$ matrix calculated as:

$$B = MR \tag{37}$$

with M as the fundamental matrix and R is the sub-matrix an the TPM in (1).

Source of Data Used

Since this research is focused on a particular department in a particular institution, the data to be used for this research will be the enrolment list of students in the Department of Mathematics, Faculty of Science in Ekiti State University, Ado–Ekiti, Nigeria, from 2010-2011 to 2015-2016. The data will be collected from the Records Department of the Ekiti State University. It is a primary data collected at the point of enrolment.

Statement Problem

The student's enrolment and their performances were taken over 6 years and to be observed. The students' performances will be determined under the following: Enrolment, Promotion, Repeated, Graduated, and Withdrawn. Also, the research will estimate the transition probabilities, calculate the fundamental matrix entries, and evaluate the absorption probability for withdrawn students, and that of the graduation. It will also compute the n −step transition probabilities necessary for the student's enrolment that will make the research findings best described as being precise, valid, unbiased, usable, and correct.

The data will be analyzed using Mat Lab, SPSS, or R software to compute the transition probabilities, and estimate the entries of the fundamental Matrix M, and determine the absorption probability by computing the $n - step$ transition probabilities.

Modeling of the Absorbing Markov Chain

Markov chain as a probabilistic model is used to model Markov Processes. It is a typical stochastic process that is finite such that the future state of the system is wholly dependent on the previous state. If the probabilities of passing across from one state to another state in a discretionary time step are known, the probabilities can be put into a square matrix form that is referred to as a Markov chain.

With the fusion of an absorbing state into an existing state coupled with the capability of a system or an object from a non-absorbing state to attain an absorbing state then, the Markov chains transit into the absorbing Markov chains. Over a period of time, everything in the system will be absorbed into the absorbing state. This is as seen in the car example by Roberts (1976). Consequently, single absorbing state Markov chains do not offer valid results while Markov chains with multiple absorbing states provide interesting and useful results. Markov chains with multiple absorbing states are left for the subsequent examples.

By the National University Commission (NUC) and the rules governing Ekiti State University Ado, Nigeria as approved by the University senate a student stands to complete the bachelor's degree program in Mathematics if such candidates have spent the minimum of four academic calendars (eight semesters), satisfied, and passed all the required courses. Else, the academic year of such a candidate stands to be more than the stipulated four academic sessions.

Therefore, to model the students' academic progress it is pertinent to define the following academic states or levels:

$1L$ - the student has been registered into the first academic year with a freshman status;
$2L$ - the student has been promoted to the second year as a sophomore;
$3L$ - the student has been promoted to the third year as a junior status or penultimate class;
$4L$ - the student has been promoted to the final year of the as a senior student in the University:
W - the student has on academic grounds been withdrawn from the University;
G - the student has completed and satisfied all the conditions for graduation.

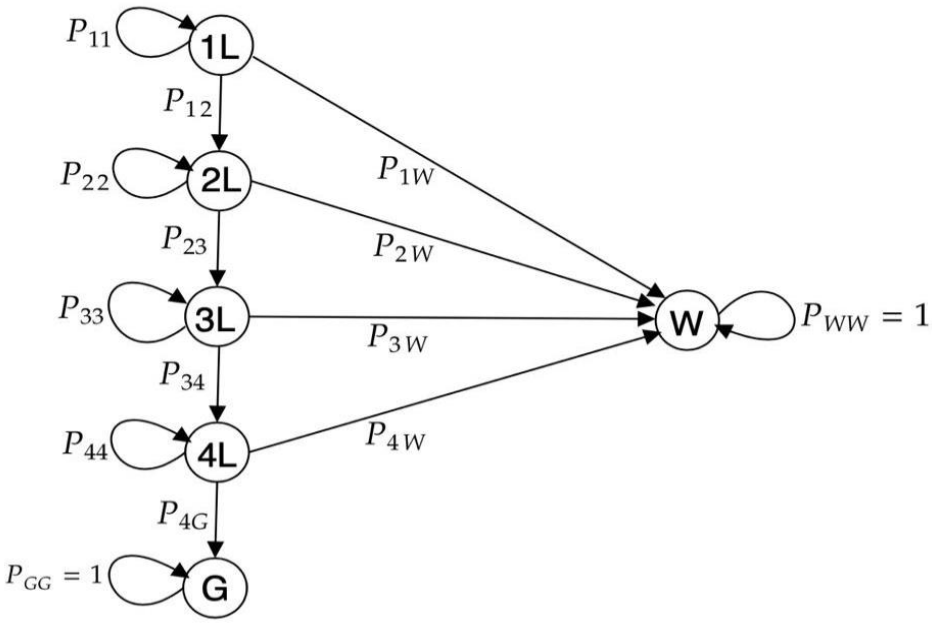

Figure 1: Transition diagram showing the students' academic progress

From Figure 1, the entries of the TPM accounted for by the Markov Chain for the students' academic program from the year of admission to the year of graduation is given as:

$$P = \begin{array}{c} \\ 1L \\ 2L \\ 3L \\ 4L \\ W \\ G \end{array} \begin{array}{c} \begin{array}{cccccc} 1L & 2L & 3L & 4L & W & G \end{array} \\ \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{1W} & P_{1G} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{2W} & P_{2G} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{3W} & P_{3G} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{4W} & P_{4G} \\ P_{W1} & P_{W2} & P_{W3} & P_{W4} & P_{WW} & P_{WG} \\ P_{G1} & P_{G2} & P_{G3} & P_{G4} & P_{GW} & P_{GG} \end{pmatrix} \end{array} \quad (38)$$

The values of the entries of the transition probability matrix are determined according to thus:

All the arcs in Figure 1 that are not possible to occur leads to transition probability matrix entry value of 0. The arcs that are certain to occur have the transition probability matrix entry of 1 while other entries have probable matrix values that have to be computed using (33) appropriately leading to:

$$P = \begin{array}{c} \\ 1L \\ 2L \\ 3L \\ 4L \\ W \\ G \end{array} \begin{array}{c} \begin{array}{cccccc} 1L & 2L & 3L & 4L & W & G \end{array} \\ \begin{pmatrix} P_{11} & P_{12} & 0 & 0 & P_{1W} & 0 \\ 0 & P_{22} & P_{23} & 0 & P_{2W} & 0 \\ 0 & 0 & P_{33} & P_{34} & P_{3W} & 0 \\ 0 & 0 & 0 & P_{44} & P_{4W} & P_{4G} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad (39)$$

Model Assumptions: The model is developed considering the following necessary assumptions:

- students who are presently in their first, second, or third academic year can either progress to the next level or repeat the present level;
- students that are present in the final academic year can either repeat the same level if such a student fails or progress to graduate from the course of study;

- any student whose studentship has been withdrawn based on academic grounds cannot complete or finish the study;
- students who have graduated and successfully finished a program cannot reapply for the same program again.

Data Smoothening

The enrolment, academic performance, and progress of students for six consecutive academic calendars are shown in Table 1. The number of enrolments in 2010, 2011, 2012, 2013, 2014, and 2015 is 185, 160, 133, 154, 117, and 119 respectively.

Table 1. Students' enrolment, academic performance, and progress

	2010 Academic Session Entrants				2011 Academic Session Entrants				2012 Academic Session Entrants			
	1st year	2nd year	3rd year	4th year	1st year	2nd year	3rd year	4th year	1st year	2nd year	3rd year	4th year
Promoted	151	143	128	0	145	143	139	0	121	120	118	0
Repeated	26	19	25	19	13	9	8	5	11	7	5	8
Withdrawn	8	15	9	5	2	6	5	3	1	5	4	1
Graduated	0	0	0	129	0	0	0	139	0	0	0	114
	2013 Academic Session Entrants				2014 Academic Session Entrants				2015 Academic Session Entrants			
	1st year	2nd year	3rd year	4th year	1st year	2nd year	3rd year	4th year	1st year	2nd year	3rd year	4th year
Promoted	134	128	139	0	106	103	98	0	97	90	87	0
Repeated	14	26	15	9	8	9	10	9	15	19	9	13
Withdrawn	6	6	0	0	3	2	4	2	7	3	3	2
Graduated	0	0	0	145	0	0	0	97	0	0	0	81

From Table 1, it is assumed that all the readings will begin in the 2010 academic session to enable proper smoothing of the data. The data shows the academic progress and performance of the students stating how many students were promoted, repeated, withdrawn, or graduated in each session. The year 2010 is taken as a point of reference such that the number of withdrawn students have to be subtracted from the number of students that have to move to the next level.

Going by the university laws and regulations, it permits any student that has to repeat the course(s) to progress to the next level but the student must have to retake/repeat the failed course(s).

Table 2 shows the cumulative enrolment and cumulative progress of students for six consecutive academic calendars.

Table 2. Students' cumulative enrolment and cumulative progress

The number of entrants in the year		End of 1^{st} year 2011	End of 2^{nd} year 2012	End of 3^{rd} year 2013	End of 4^{th} year 2014	End of 5^{th} year 2015	End of 6^{th} year 2016	End of 7^{th} year 2017	End of 8^{th} year 2018	End of 9^{th} year 2019	End of 10^{th} year 2020
2010	185	177	162	153	**129**	**8**	**5**	**6**	***Nil***	***Nil***	***Nil***
2011 2010	*Nil*	160	158	152	147	**139** 19	**2** 11	**3** 6	***Nil***	***Nil***	***Nil***
2012 2011 2010	*Nil*	*Nil*	133	132	127	123	**114** 5 11	**6** 3 6	**2** **2** ***Nil***	***Nil*** ***Nil*** ***Nil***	***Nil*** ***Nil*** ***Nil***
2013 2012 2011 2010	*Nil*	*Nil*	*Nil*	154	148	154	154	**145** 8 3 6	**5** **3** **1** ***Nil***	**1** **2** ***Nil*** ***Nil***	***Nil*** **2** ***Nil*** ***Nil***
2014 2013 2012 2011 2010	*Nil*	*Nil*	*Nil*	*Nil*	117	114	112	108	**97** **6** **2** ***Nil*** ***Nil***	**5** **1** ***Nil*** ***Nil*** ***Nil***	***Nil*** ***Nil*** ***Nil*** ***Nil*** ***Nil***
2015 2014 2013 2012 2011 2010	*Nil*	*Nil*	*Nil*	*Nil*	*Nil*	119	112	109	96	**81** 5 8 ***Nil*** ***Nil*** ***Nil***	**10** ***Nil*** ***Nil*** ***Nil*** ***Nil*** ***Nil***

From Table 2, it is assumed that all the reading will begin in the 2010 academic session and the cumulative of repeated for the previous year is added to the enrolment of the present session but with different session readings. The data shows the academic stand of the students over the six sessions and three graduating years were recorded in 2013, 2014, and 2015 for students that were admitted in 2010, 2011, and 2012 respectively. Herein, the enrolment less the candidates that have been withdrawn and those to repeat gives the number of students that will have to move to the next level designated as promoted.

Transition Probability Matrix Computation

Below are the students' academic progress records over four consecutive years. This table will be used to determine and evaluate the TPM and others requisite parameters.

Table 3. Students' academic progress records

	At the end of the 1^{st} academic year	At the end of the 2^{nd} academic year	At the end of the 3^{rd} academic year	At the end of the 4^{th} academic year
Promoted	103	87	75	0
Repeated	55	22	12	5
Withdrawn	16	12	7	8
Graduated	0	0	0	62

From Table 3 the transition matrix is obtain and it will guide in the computation of the Transition Probability Matrix.

$$n_{ij} = \begin{array}{c} \cdot \\ 1L \\ 2L \\ 3L \\ 4L \\ W \\ G \end{array} \begin{array}{c} \begin{array}{cccccc} 1L & 2L & 3L & 4L & W & G \end{array} \\ \begin{pmatrix} 55 & 103 & 0 & 0 & 16 & 0 \\ 0 & 22 & 87 & 0 & 12 & 0 \\ 0 & 0 & 12 & 75 & 7 & 0 \\ 0 & 0 & 0 & 5 & 8 & 62 \\ 0 & 0 & 0 & 0 & 43 & 0 \\ 0 & 0 & 0 & 0 & 0 & 62 \end{pmatrix} \end{array} \quad (40)$$

Note that the Transition matrix does not allow for cumulating entrants from different years. However, it is meant to compute progression over the entire academic period with the entries along the diagonal being the students that repeat. The entries below the diagonal are all zero showing that the students have not been on the list or admitted into the institution as at that time while other entries that are zeros above the diagonal are the entries that are not possible to occur. For example, a student cannot progress directly from 100level to 300level neither is it possible to progress directly from 200level to 400level nor is it possible to graduate from 100level, 200level or 300level irrespective of the student's academic brilliance hence, the entries are zeros.

The table shows the enrolment and performance summary of the 724 students.

Table 4. Students' enrolment and performance summary

	1st Session	2nd Session	3rd Session	4th Session	Total
Promoted	213	195	207	0	615
Repeated	455	422	400	225	225
Withdrawn	56	32	17	15	120
Graduated	0	0	0	360	360
Total	724	649	624	600	

From Table 4, the one-step transition matrix is as follows:

$$n_{ij} = \begin{array}{c} \cdot \\ 1L \\ 2L \\ 3L \\ 4L \\ W \\ G \end{array} \begin{array}{c} \begin{array}{cccccc} 1L & 2L & 3L & 4L & W & G \end{array} \\ \begin{pmatrix} 455 & 213 & 0 & 0 & 56 & 0 \\ 0 & 422 & 195 & 0 & 32 & 0 \\ 0 & 0 & 400 & 207 & 17 & 0 \\ 0 & 0 & 0 & 225 & 15 & 360 \\ 0 & 0 & 0 & 0 & 120 & 0 \\ 0 & 0 & 0 & 0 & 0 & 360 \end{pmatrix} \end{array} \quad (41)$$

To determine the transition probability matrix in (41), each entry on the row must be divided by the sum of all the entries on the row. Hence, the transition probability matrix arrived at will be in a canonical form. The canonical matrix unwraps the fact that the probability that a student will remain at the same level as well as the possibility of the student progressing to the next level.

$$P_{ij} = \begin{pmatrix} 0.6285 & 0.2942 & 0.0000 & 0.0000 & 0.0773 & 0.0000 \\ 0.0000 & 0.6502 & 0.3005 & 0.0000 & 0.0493 & 0.0000 \\ 0.0000 & 0.0000 & 0.6410 & 0.3317 & 0.0272 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.3750 & 0.0250 & 0.6000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \quad (42)$$

The one-step TPM, P_{ij}, in (42) is then transformed into a block matrix as in (1) such that Q a $n \times n$ matrix is the transient state matrix is given as:

$$Q = \begin{pmatrix} 0.6285 & 0.2942 & 0.0000 & 0.0000 \\ 0.0000 & 0.6502 & 0.3005 & 0.0000 \\ 0.0000 & 0.0000 & 0.6410 & 0.3317 \\ 0.0000 & 0.0000 & 0.0000 & 0.3750 \end{pmatrix} \quad (43)$$

R a $n \times r$ matrix represents the absorbing state matrix that is:

$$R = \begin{pmatrix} 0.0730 & 0.0000 \\ 0.0493 & 0.0000 \\ 0.0272 & 0.0000 \\ 0.0250 & 0.6000 \end{pmatrix} \tag{44}$$

I stands for an $r \times r$ identity matrix given as:

$$I = \begin{pmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{pmatrix} \tag{45}$$

and 0 stands for the $r \times n$ zero matrix:

$$0 = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix} \tag{46}$$

Two or more of the matrices $Q, R, I,$ and O are put together to determine the requisite matrices needed for further estimations as it will be shown in the next sub-sections.

Fundamental Matrix Computation

The fundamental matrix given by $M = (I - Q)^{-1}$ is such that I herein has an order different from the I from the block matrices given as:

$$M = \left(\begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} - \begin{pmatrix} 0.6285 & 0.2942 & 0.0000 & 0.0000 \\ 0.0000 & 0.6502 & 0.3005 & 0.0000 \\ 0.0000 & 0.0000 & 0.6410 & 0.3317 \\ 0.0000 & 0.0000 & 0.0000 & 0.3750 \end{pmatrix} \right)^{-1}$$

$$M = \begin{pmatrix} 0.3715 & -0.2942 & 0.0000 & 0.0000 \\ 0.0000 & 0.3498 & -0.3005 & 0.0000 \\ 0.0000 & 0.0000 & 0.3590 & -0.3317 \\ 0.0000 & 0.0000 & 0.0000 & 0.6250 \end{pmatrix}^{-1}$$

$$M = \begin{pmatrix} 1.5911 & -1.3382 & 0.3375 & -0.0299 \\ 0.0000 & 1.5338 & -0.7211 & 0.6378 \\ 0.0000 & 0.0000 & 1.5601 & -1.3799 \\ 0.0000 & 0.0000 & 0.0000 & 2.6667 \end{pmatrix} \tag{47}$$

From (47), it is obvious that from the point of enrolment into the program, it is expected that a student will spend 1.5911 for the first academic year, 1.5338 for the second academic year, 1.5601 for the third academic year, and 2.6667 for the fourth academic year especially for all study programs designated for four academic calendars.

Expected Time Until Absorption for Withdrawal or Graduation

With a recall to the absorption

$$\mu = M \times C$$

$$\mu = \begin{pmatrix} 1.5911 & -1.3382 & 0.3375 & -0.0299 \\ 0.0000 & 1.5338 & -0.7211 & 0.6378 \\ 0.0000 & 0.0000 & 1.5601 & -1.3799 \\ 0.0000 & 0.0000 & 0.0000 & 2.6667 \end{pmatrix} \times \begin{pmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 0.5605 \\ 1.4505 \\ 0.1802 \\ 2.6667 \end{pmatrix} \tag{48}$$

The result on the right-hand side of (48) is the required time for absorption (Graduation or withdrawal) from a given transient state that can be attained. It indicates that it will take

$0.5605, 1.4505, 0.1802,$ and 2.6667 expected time for the first, second, third, and fourth years respectively before the necessary absorption for graduation to be effected from the given transient state.

Absorption Probability for Withdrawal or Graduation

The probability that a system will enter the jth absorbing state if it begins from the ith transient state is called the probability of absorption. This is given as:

$$B = M \times R$$

$$B = \begin{pmatrix} 1.5911 & -1.3382 & 0.3375 & -0.0299 \\ 0.0000 & 1.5338 & -0.7211 & 0.6378 \\ 0.0000 & 0.0000 & 1.5601 & -1.3799 \\ 0.0000 & 0.0000 & 0.0000 & 2.6667 \end{pmatrix} \times \begin{pmatrix} 0.0730 & 0.0000 \\ 0.0493 & 0.0000 \\ 0.0272 & 0.0000 \\ 0.0250 & 0.6000 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.05860954 & 0.00000 \\ 0.07194742 & 0.00000 \\ 0.00793722 & 0.00000 \\ 0.06667500 & 1.60002 \end{pmatrix} \tag{49}$$

From B in (49), the first column stands for the withdrawal status from the first, second, third, and fourth academic session in that order while the second column stands for the graduating students from the first to the fourth academic year in that order.

From (49), the probability of graduating in the first, second, and third year is zero meaning that there is no how a student could have graduated from the program since it is a four-year minimum program while the probability that a student might withdraw will either increase or decrease as the case may be.

Conclusion

Though, since humans are involved such that every individual is psychologically, socially, emotionally, and financially affected in different ways, the academic performance cannot be the same for two different people more so that they have different behavioral responses to learning situations.

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