Khintchine and Rosenthal Inequalities in Non-Commutative Symmetric Banach Functions Spaces

[1]Abubaker Mukhtatar I., [2]Hail Suwayhi. A., [3]Ibrahim Hamza H.
[1]Math. PhD, Alimam Alhadi College, Omdurman, Sudan
[2]Math. MSc, Saudi Arabia Ministry of Education, Saudi Arabia
[3]Math. PhD, Omdurman Islamic University, Omdurman, Sudan

Abstract. We present a direct proof of the upper Khintchine inequality for non-commutative symmetric spaces and derive an improved version of Rosenthal Theorem for sums of independent random variables in non-commutative symmetric spaces. As a result we obtain a new proof of Rosenthal Theorem for L^p spaces.

Keywords: Symmetric spaces, non-commutative spaces, Rosenthal inequalities, Khintchine inequalities

I. Introduction and Preliminaries

We introduce and extending several inequalities to the field of non-commutative symmetric Banch function spaces. We generalize some classical inequalities for independent random variables, due to H. P. Rothenthal. Rosenthal inequality (Astoshkin & Maligranda, 2004) was initially discovered to construct some new Banach spaces. However, Rosenthal inequality gives a good bound for the $p-norm$ of independent random variables and has found many generalizations and applications.

The classical Rosenthal inequality (Montgmery-Smith, n.d.; Theorem3) assert that for $p \geq 2$ and (x_i) a sequence of independent, mean zero random variable in $L^p(\Omega)$, where $(\Omega, x, \mathbb{P})$ is a probability space, we has

$$\|\sum_{i=1}^{n} x_i\|_{L^p(\Omega)} \simeq_p max\left\{\|(x_i)_{i=1}^{n}\|_{L_p(L_p(\Omega))}, \|(x)_{i=1}^{n}\|_{L^2.\left(L^2(\Omega)\right)}\right\}, \quad (1)$$

1.1. Definition. Banach function space $E\ on\ (0,\alpha)$ is called symmetric if for $f \in S(0,\alpha)\ and\ g \in E\ with\ \mu(f) \leq \mu(g)$ we have $f \in E$ and $\|f\|_E \leq \|g\|_E$.

Let $(\mathcal{N},\tau)$ be a semifinite von Neuman algebra then we can state the following definitions.

1.2. Definition. For $\tau - measurabe$ operator x (affiliated with $\mathcal{N}$) $and\ t > 0$, the singular value of x is defined by:

$$\mu_t(x) := inf\left\{S \geq 0 : \tau\left(\mathcal{X}_{[S,\infty)}(|x|)\right) \leq t\right\}.$$

An equivalent formulation $\mu_t(x) = inf\{\|x(1-p)\|_\infty : \tau(p) \leq t\}$.

1.3. Definition. Given a symmetric function space E on $(0,\infty), set\ E(\mathcal{N},\tau) \coloneqq \{x; \mu(x) \in E\}$. Then $E(\mathcal{N},\tau)$ is called the noncommutative symmetric space associated to $\mathcal{N}\ and\ E$.

Now we describe a special case of the main result (Theorem 4.4 in the following section 4). Let E be a separable symmetric Banach function space on $(0,\infty)$ and $\mathcal{M}$ be a semi finite von Neumann algebra. We denote by $E(\mathcal{M})$ the non commutative symmetric space associated with $E\ and\ \mathcal{M}$. Let $(\mathcal{N}_k)$ be a sequence of von Neumann subalgebra of the $\mathcal{M}$, $\mathcal{N}$ a common von Neumann sub algebra of the $(\mathcal{N}_k)$ and suppose that $(\mathcal{N}_k)$ is independent with respect to $\mathcal{E}_\mathcal{N}$, the conditional expectation with respect to $\mathcal{N}$. Let (x_k) be a sequence such that $x_k \in E(\mathcal{N}_k)\ and\ \mathcal{E}_\mathcal{N}(x_k) = 0$ for all k. Let $diag(x_k)$ denote the $n \times n$ diagonal matrix with $x_1, \dots, x_n$ on its diagonal. Then for any n

$$\|\sum_{k=1}^{n} x_k\|_{E(\mathcal{M})} \simeq_E max\left\{\|diag(x_k)_{k=1}^{n}\|_{E(\mathcal{M}_n(\mathcal{M}))}, \left\|(\sum_{k=1}^{n} \varepsilon_\mathcal{N}|x_k^*|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}\right\}, \quad (2)$$

provided E satisfies one of the following conditions:

(i) either E has Boyd indices satisfying $2 < p_E \leq q_E < \infty$,

(ii) E is a symmetric Banach function space which is an interpolation space for couple (L^2, L^p) and $q - concave$, for some $2 \leq p < \infty$ and $q < \infty$.

Corresponding to the two conditions defined above, we need two different types of Khintchine inequalities in the proof of (2). Under condition (i), a key tool needed in proving the above generalization of Rosenthal's Theorem is the following Khintchine type inequality, considered in (Lemardy & Sukochev, 2008). Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Suppose E is a symmetric Banach function space on $(0,\infty)$ $with$ $1 < p_E \leq q_E < \infty$ which is either separable or the dual of separable space. The main result in (Lemardy & Sukochev, 2008). Theorem1.1 states that for any finite sequence (x_k) in $E(\mathcal{M})$ and any Rademacher sequence $(r_k)_{k=1}^{\infty}$ we have

$$\|\textstyle\sum_k r_k \otimes x_k\|_{E(L^\infty(\Omega)\bar{\otimes}\mathcal{M})} \lesssim_E max\left\{\|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|x_k\|_{E(\mathcal{M};l_r^2)}\right\}. \quad (3)$$

This inequality is derived by duality form the following inequality, which holds for a larger class of spaces: if $q_E < \infty$, then

$$inf\left\{\|y_k\|_{E(\mathcal{M};l_c^2)} + \|(z_k)\|_{E(\mathcal{M};l_r^2)}\right\} \lesssim E\|r_k \otimes x_k\|_{E(L^\infty(\Omega)\bar{\otimes}\mathcal{M})},$$

where the infimum is taken over all decompositions $x_k = y_k + z_k$. In the work by Astashkin (2010), it was left as an open question whether (3) holds if $q_E < \infty$, with no further assumption on the lower Boyed index of E. We answer this question in the positive by giving a direct proof in the following 4.1 Theorem of section 4 . In fact, we obtain (3) also for quasi-Banach function spaces and this proves to be essential for the proof of (2).

For any finite sequence (x_k) in $E(\mathcal{M})$ and Rademacher sequence $(r_k)_{k=1}^{\infty}$ we have

$$E\|\textstyle\sum_k r_k x_k\|_{E(\mathcal{M})} \lesssim max\left\{\|(x_k)\|_{E(\mathcal{M});l_c^2}, \|(x_k)\|_{E(\mathcal{M});l_r^2}\right\}. \quad (4)$$

II. Symmetric Banach Function Spaces

Let $0 < \alpha \leq \infty$. For a measurable , finite function f on $(0,\alpha)$ we define its distribution function by $d(v;f) = \lambda(t \in (0,\alpha): |f(t)| > v\,)$ for $v > 0$, $where$ λ denotes Lebesgue measure. For $f, g \in (0,\alpha)$ we say f is submajorized by g and we write $f \prec\prec g$, if $\int_0^t \mu_s(f)ds \leq \int_0^t \mu_s(g)ds$ for all $t > 0$. A quasi-Banach function space E on $(0,\infty)$ is called symmetric if for all $\in S(0,\alpha)$ and $g \in E$ $with$ $\mu(f) \leq \mu(g)$ we $have$ $\|f\|_E \leq \|g\|_E$. It is called strongly symmetric if , in addition, for $f, g \in E$ $with$ $f \prec\prec g$ we have $\|f\|_E \leq \|g\|_E$. If , moreover, for $f \in S(0,\alpha)$ and $g \in E$ with $f \prec\prec g$ it is follows that $f \in E$ and $\|f\|_E \leq \|g\|_E$, then E is called fully symmetric. Fully symmetric Banach function spaces on $(0,\alpha)$ are exact interpolation spaces for the couple $\left(L^1(0,\alpha), L^\infty(0,\alpha)\right)$. The class of (fully) symmetric spaces covers many interesting spaces from harmonic a analysis and interpolation theory, which as Lorentz, Marcinkiewicz and Orlicz spaces.

A symmetric (quasi-Banach function space is said to have a Fatou quasi-norm if for every net $\left(f_\beta\right)$ in E and $f \in E$ satisfying $0 \leq f_\beta \uparrow f$ we have $\left\|f_\beta\right\|_E \uparrow \|f\|_E$. The space E is said to have the Fatou property if for every $\left(f_\beta\right)$ in E and $f \in S(0,\alpha)$ satisfying $0 \leq f_\beta \uparrow f$ and $sup_\beta\left\|f_\beta\right\|_E < \infty$ we have $f \in E$ and $\left\|f_\beta\right\|_E \uparrow \|f\|_E$.

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful trace state $\tau: \mathcal{M} \to \mathbb{C}$, $that$ is $\tau(1) = 1$ and $\tau(xy) = \tau(yx)$. Then $L_p(\mathcal{M},\tau)$ is completion of $\mathcal{M}$ with respect to $\|x\|_p = [\tau|x|^p]^{1/p}$. It is well known (Johnson & Schechtman, 1988; Takesaki, 1972) that $\|.\|_p$ is a norm for $1 \leq p \leq \infty$. In particular $\|.\|_\infty = \|.\|$. $\|.\|$ denote operator norm . Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra . Then there exist a unique conditional expectation $\mathcal{E}_{\mathrm{N}}: \mathcal{M} \to \mathcal{N}$ such that $\mathcal{E}_{\mathcal{N}}(1) = 1$ and $\mathcal{E}_{\mathcal{N}}(axb) = a\mathcal{E}_{\mathcal{N}}(x)b$, $a, b \in \mathcal{N}$ and $\in \mathcal{M}$. We say that two subalgebras $\mathcal{N} \subset A, B \subset \mathcal{M}$ are independent over $\mathcal{N}$ if $\mathcal{E}_{\mathcal{N}}(ab) = \mathcal{E}_{\mathcal{N}}(a)\mathcal{E}_{\mathcal{N}}(b), a \in A, b \in B$.

For any $x \in L^1 + L^\infty(\mathcal{M})$, $\mathcal{E}(x)$ is the unique element in $L^1 + L^\infty(\mathcal{N})$ satisfying

$$\tau(xy) = \tau(\mathcal{E}(x)y), \text{for all} \in L^1 \cap L^\infty(\mathcal{N}). \quad (5)$$

The $\mathcal{E}$ is called conditional expectation with respect to the von Neumann sub algebra $\mathcal{N}$.

2.1. Proposition (Johnson & Schechtman, 1988). Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ and let $\mathcal{N}$ be a von Neumann subalgebra such that restriction of τ *to* $\mathcal{N}$ a gain semi-finite. Then there is a unique linear map $\mathcal{E}: L^1 + L^\infty(\mathcal{M}) \to L^1 + L^\infty(\mathcal{N})$ satisfying the following properties :

(a) $\mathcal{E}(x^*) = \mathcal{E}(x)^*$;

(b) $\mathcal{E}(x) \geq 0 \; if \; x \geq 0$;

(c) $if \; x \geq 0$ and $\mathcal{E}(x) = 0 \; then \; x = 0$;

(d) $\mathcal{E}(x) = x \; for \; any \; x \in L^1 + L^\infty(\mathcal{N})$;

(e)$\mathcal{E}(x) * \mathcal{E}(x) \leq \mathcal{E}(x * x) \; for \; x \in \mathcal{M}$;

(f) $\mathcal{E}$ is normal , i.e. $x_\alpha \uparrow x \; implies \; \mathcal{E}(x_\alpha) \uparrow \mathcal{E}(x) \; for \; (x_\alpha), x \in \mathcal{M}$;

(g)$\|\mathcal{E}(x)\|_1 \leq \|x\|_1, for \; all \; x \in L^1(\mathcal{M}), \|\mathcal{E}(x)\|_\infty \leq \|x\|_\infty$, for all $x \in \mathcal{N}$ and so $\mathcal{E}(x) \prec\prec x$ for all $\in L^1 + L^\infty(\mathcal{M})$;

(h)$\mathcal{E}(xy) = x\mathcal{E}(y) \;\; if \;\; x \in L^1(\mathcal{N}), \;\; y \in L^\infty(\mathcal{M}) \;\; and \;\; \mathcal{E}(xy) = \mathcal{E}(x)y$ whenever $\in L^1(\mathcal{M}), y \in L^\infty(\mathcal{N})$.

2.2. Definition. Let $0 < \alpha \leq \infty \; and \; let \; E$ be a symmetric quasi-Banach function space on $(0, \alpha)$. For any $0 < \alpha < \infty$ we define the dilation operator $D_n \; on \; S(0, \alpha)$ by $(D_n f)(s) = f(ns)\chi_{(0,\alpha)}(ns)$.

If E is a symmetric quasi-Banach function space on $(0, \alpha)$, then D_n is a bounded linear operator.

Define the lower Boyd index p_E of E by

$$p_E = sup\left\{p > 0 : \; \exists \; c > 0 \,, \forall 0 < n \leq 1 \; \|D_n f\|_E \leq c n^{\frac{-1}{p}} \|f\|_E\right\},$$

and the upper Boyd index by

$$q_E = inf\left\{q > 0 : \exists c > 0 \forall n \geq 1 \; \|D_n f\|_E \leq c n^{\frac{-1}{q}} \|f\|_E \;\right\}.$$

This above two definitions of lower and upper Boyd index can be denoted by

$$p_E = \lim_{s \to \infty} \frac{logs}{log\left\|D_{1/s}\right\|} = \sup_{s > 1} \frac{logs}{log\left\|D_{1/s}\right\|},$$

$$q_E = \lim_{s \to 0} \frac{logs}{log\left\|D_{1/s}\right\|} = \inf_{0 < s < 1} \frac{logs}{log\left\|D_{1/s}\right\|} .$$

Note that $0 \leq p_E \leq q_E \leq \infty$. If E is a symmetric Banach function space then $1 \leq p_E \leq q_E \leq \infty$. Let $0 < p_E \,, q_E \leq \infty$. A symmetric quasi-Banach function space E is said to be $p - convex$ if there exists a constant $c > 0$ such that for any finite sequence $(x_i)_{i=1}^n$,

$$\left\|\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}\right\|_E \leq c\left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}}, \quad (if \; 0 < p < \infty) \,,$$

$$\|\max_{1 \leq i \leq n} |x_i|\|_E \leq c \max_{1 \leq i \leq n} \|x_i\|_E \quad (if \; p = \infty) \,.$$

A symmetric quasi-Banach function space E is said to be q-concave if there exists a constant $c > 0$ such that for any finite sequence $(x_i)_{i=1}^n$ in E we have $\left(\sum_{i=1}^n \|x_i\|_E^q\right)^{\frac{1}{q}} \leq c\left\|\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}\right\|_E , (if \; 0 < q < \infty),$

$$\max_{1 \leq i \leq n} \|x_i\|_E \leq c \|\max_{1 \leq i \leq n} |x_i|\|_E \;\; (if \;\; q = \infty) \,.$$

We can conclude that every quasi-Banach function is $\infty - concave$ and any Banach function space is $1 - convex$. We note that if $E \; is \; p - convex$ then $p_E \geq p \; and \; if \; E \; is \; q -$

$concave\ then\ q_E \leq q$. For $1 \leq r < \infty$, let the r-concavification and r-convexification of E be defined by $E_{(r)} := \{g \in S(0,\alpha): |g| = f^r; f \in E\}, \|g\|_{E_{(r)}} = \|f\|_E^r$

$E^{(r)} := \left\{g \in S(0,\alpha): |g| = f^{\frac{1}{r}}; f \in E\right\}, \|g\|_{E^{(r)}} = \|f\|_E^{\frac{1}{r}}$, respectively (Lindenstrauss & Tzafriri, 1979) (p.53), if E is a (symmetric) Banach function space, then $E^{(r)}$ is a (symmetric) Banach function space. From the above definitions it follows that $p_{E_{(r)}} = \frac{1}{r} p_E$, $q_{E_{(r)}} = \frac{1}{r} q_E$, $p_{E^{(r)}} = r p_E$, $q_{E^{(r)}} = r q_E$.

2.3. Lemma (Johnson & Schechtman, 1988). Let E be a symmetric quasi-Banach function space. Then for every a constant $p > 0$ there exists a constant $c > 0$ and $0 < r \leq p$ such that for all $x_i \in E$

$$\left\|\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}\right\|_E \leq c\left(\sum_{i=1}^{\infty} \|x_i\|_E^r\right)^{\frac{1}{r}}. \tag{6}$$

For $f \in (0,\alpha)$ we set

$K(t,f;L^p,L^q) = inf\left(\|f_0\|_{L^p(0,\alpha)} + \|f_1\|_{L^q(0,\alpha)}\right) f = f_0 + f_1$, $(t > 0)$.

2.4. Theorem. Let $1 \leq p < q \leq \infty$ and $\frac{1}{p'} + \frac{1}{p} = 1, \frac{1}{q'} + \frac{1}{q} = 1$. Suppose E is a separable Banach-function space on $(0,\alpha)$ and suppose $E^{\times}$ is an interpolation space for the couple $\left(L^{p'}(0,\alpha), L^{q'}(0,\alpha)\right)$. Then E is an interpolation space for the couple $\left(L^p(0,\alpha), L^q(0,\alpha)\right)$.

2.5. Theorem. Let (S,Σ,μ) be a measure space and let $1 \leq p,q \leq \infty$. Then every interpolation space E for the couple $\left(L^p(S), L^q(S)\right)$ is given by a k-method i.e., there is a Banach function space f on $(0,\infty)$ such that $f \in E$ if and only if $t \to K(t,f;L^p,L^q) \in f$ and there exist constants $c, C > 0$ such that

$c\|t \to K(t,f;L^p,L^q)\|_f \leq \|f\|_E \leq C\|T \mapsto K(t,f;L^p,L^q)\|_F$.

2.6. Proposition. Let $1 \leq p \leq q \leq \infty$. Suppose E is a symmetric quasi- Banach function space on $(0,\alpha)$ which is an interpolation space for couple $\left(L^p(0,\alpha), L^q(0,\alpha)\right)$. Then for any $0 \leq s-1 < \infty$, $E_{(s-1)}$ respectively $E^{(s-1)}$ is an interpolation space for the couple $\left(L^{\frac{p}{s-1}}(0,\alpha), L^{\frac{q}{s-1}}\right)$ respectively $\left(L^{p(s-1)}(0,\alpha), L^{q(s-1)}(0,\alpha)\right)$.

Proof. By Theorem 2.5. there exists a Banach function space F such that

$\|f\|_E = \|t \longmapsto K(t,f;L^p,L^q)\|_F$.

For any $p > 0$ and $a,b \geq 0, \alpha_p(a^p + b^p) \leq (a+b)^p \leq \beta_p(a^p + b^p)$, for some constants α_p, β_p depending only on p. Using this fact, it is not difficult to show that there are constants depending only on s such that

$$K\left(t, |f|^{\frac{1}{s-1}}; L^p, L^q\right) \simeq K\left(t^{s-1}, f; L^{\frac{p}{s-1}}, L^{\frac{q}{s-1}}\right).$$

Let T be a linear operator on $L^{\frac{p}{s-1}} + L^{\frac{q}{s-1}}$ which is bounded on $L^{\frac{p}{s-1}}$ and $L^{\frac{q}{s-1}}$. Then

$$\|Tf\|_{E_{(s-1)}} \simeq \left\|t \longmapsto K\left(t^{s-1}, Tf; L^{\frac{p}{s-1}}, L^{\frac{q}{s-1}}\right)^{\frac{1}{s-1}}\right\|_F \lesssim \left\|t \mapsto K\left(t^{s-1}, f; L^{\frac{p}{s-1}}, L^{\frac{q}{s-1}}\right)^{\frac{1}{s-1}}\right\|_F \simeq \|f\|_{E_{(s-1)}}.$$

We can similarly prove the assertion for $E^{(s-1)}$.

2.7. Lemma. Let E be a symmetric quasi-Banach space on $(0,\alpha)$ $with$ $p_E > 0$. Then E is an interpolation space for the couple $\left(L^p(0,\alpha), L^q(0,\alpha)\right)$.

Proof. Suppose first $p_E > 1$. We claim that E is fully symmetric up to a constants i.e., there is a constant $c_E > 0$ depending only on E, such that if $f \in S(0,\alpha), g \in E$ and $f \prec\prec g$, then $f \in E$ and $\|f\|_E \leq c_E\|g\|_E$. Let $g^{**}(t) = \frac{1}{t}\int_0^t \mu_s(g)ds$ be the Hardy –little wood maximal function of g. By (Johnson & Schechtman, 1988), Theorem (2i), the map $g \to g^{**}$ is abounded quasi-linear map on E and $\|g^{**}\|_E \leq c_E\|g\|_E$. By assumption $f^{**} \leq g^{**}$, so $f^{**} \in E$ and $\|f^{**}\|_E \leq \|g^{**}\|_E, as\ E$ is symmetric . Finally , $\mu(f) \leq f^{**}, so\ f \in E$ and $\|f\|_E = \|\mu(f)\|_E \leq \|f^{**}\|_E$. This completes the proof of the claim.

2.8. Theorem (Kalton & Montgonetry-Smith, 2003). Let E be a symmetric quasi-Banach function space on $(0,\alpha)$ which either has order continuous quasi-norm or has the Fatou property. Let $0 < p < q \leq \infty$. Then E is an interpolation space for the couple $\left(L^p(0,\alpha), L^q(0,\alpha)\right)$ whenever one of the following conditions holds:

(i) $p < p_E \leq q_E < q$;

(ii) E is p-convex with convexity equal to 1, for some $q_E > p$;

(iii) E is $r - conex$ with convexity constant equal to 1, for some $0 < r < \infty, E$ is $q - concave$ with concavity constant equal to 1 and $p_E > p$;

(iv) E is $p - convex$ with convexity constant equal to 1 and $q - concave$.

Proof. The first assertion is the well known Boyd interpolation theorem which was generalized to symmetric quasi-Banach function spaces in (Johnson & Schechtman, 1988), theorem3. To prove the second assertion suppose first that $p \geq 1$. Then $E_{(p)}$ is a symmetric Banach function space and satisfies $q_{E_{(p)}} < \frac{q}{p}$. Moreover, $E_{(p)}$ has the Fatou property or is separable if E is. By (Astoshkin & Maligranda, 2004), the theorem1, $E_{(p)}$ is an interpolation space for the couple $\left(L^1, L^{\frac{q}{p}}\right)$. By proposition 2.6 we now find that E is an interpolation space for the couple (L^p, L^q). Finally , if $0 < p < 1$ then we find by the above that $E^{\left(\frac{1}{p}\right)}$ is an interpolation space for the couple $\left(L^1, L^{\frac{q}{p}}\right)$. Hence , by proposition 2.6, E is an interpolation space for the couple (L^p, L^q). The third assertion for $q = \infty$ is proved in Lemma 2.7. For the remaining cases we may assume , by proposition 2.6 , that $r = 1$. Under this assumption, E is a symmetric Banach function space and hence we can deduce the result by duality. Observe that E is separable , as $q < \infty$. Moreover, $E^{\times}\ is\ q' - convex$ and $q_{E^{\times}} < p'\ , where\ \frac{1}{p'} + \frac{1}{p} = 1$, $\frac{1}{q'} + \frac{1}{q} = 1$. By the second assertion we obtain that $E^{\times}$ is an interpolation space for the couple $\left(L^{q'}, L^{p'}\right)$. The result now follows from Theorem 2.4. For the final assertion, suppose first that $p = 1, q = \infty$. Then E is a symmetric Babach function space which is separable or has the Fatou property. Hence E is fully symmetric under these assumptions and the result now well-known Calderon-Mitjagin theorem, (Kreĭn, Petunin, & Semenov, 1982), theorem 4.3. The case where $p > 1, q = \infty$ follows from this by proposition 2.6. If $p = 1,\ q < \infty\ , then\ E$ is a separable symmetric Banach function space and in this case the result can be deduced from the case $p > 1, q = \infty$ by duality using theorem 2.4 in the interpolation space for the couple $\left(L^1, L^{\frac{q}{p}}\right)$. Therefore, by proposition 2.6, E is an interpolation space for the couple (L^p, L^q).

2.9. Theorem. For $1 \leq p < q \leq \infty$. Suppose E is a fully symmetric quasi-Banach function space on $(0,\alpha)$ which is an interpolation space for the couple $\left(L^p(0,\alpha), L^q(0,\alpha)\right)$. Let M be a semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(1) = \alpha$. Then $E(\mathrm{M})$ is an interpolation space for the couple $\left(L^p(\mathrm{M}), L^q(\mathrm{M})\right)$.

Let (r_k) be Rademacher sequence of independent $\{-1,1\} - valued$ random variables on a probability space $(\Omega, \mathrm{F}, \mathbb{P})$ such $P(r_k = 1) = P(r_k = -1) = \frac{1}{2}$ for all k. Then for $1 < p < \infty$, then nth Rademacher projection

$$R_n(x) = \sum_{j=1}^{n} r_j \otimes \mathcal{E}_{c_1 \bar{\otimes} \mathrm{M}}\left(\left(r_j \otimes 1\right)x\right) \quad (7)$$

is bounded on $L^p(L^\infty(\Omega)\bar{\otimes}\mathrm{M})$ and , moreover , for all $n \geq 1$ we have $\|R_n\| \leq c_p$, for some constant c_p depending only on p. if E is a symmetric quasi-Banach function space on $(0,\alpha)$ with $1 < p_E \leq q_E < \infty$, we find by interpolation that R_n defines a bounded projection in $E(L^\infty(\Omega)\bar{\otimes}\mathrm{M})$ and $\|R_n\| \leq c_E$ for some $n \geq 1$, where c_E is a constant depending only on E. We let $Rad_n(E)$ denote the image of R_n.

III. Non-Commutative Khintchine Inequalities

We prove two types of non commutative Khintchine inequalities for non commutative symmetric spaces the main results in this section are Theorem 3.1 and 3.11 below. Recall the notation

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)} = \left\|\left(\sum_i x_i^* x_i\right)^{\frac{1}{2}}\right\|_{E(\mathcal{M};l_r^2)} ; \|(x_i)\|_{E(\mathcal{M};l_r^2)} = \left\|\left(\sum_i x_i^* x_i\right)^{\frac{1}{2}}\right\|_{E(\mathcal{M})},$$

for a finite sequence (x_i) *in* $E(\mathcal{M})$.

A normal faithful, semi-finite trace τ satisfying $\tau(1) = \alpha$. Suppose E is a symmetric quasi-Banach function space on $(0,\alpha)$ which is p-convex for some $0 < p < \infty$ and satisfies $q_E < \infty$. Then

$$\left\|\sum_i r_i \otimes x_i\right\|_{E(L^\infty \bar{\otimes} \mathcal{M})} \lesssim_E max\left\{\|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)}\right\}, \quad (8)$$

for any Rademacher sequence (r_i) and any finite sequence (x_i) in $E(\mathcal{M})$.

3.1. Theorem (Lust-Piguard & Pisier, 1991). Let $1 \leq q < \infty$ and let M be a von Neumann algebra equipped with a normal, faithful, semi-finite trace . If $2 \leq q < \infty$ then

$$E\left\|\sum_i r_i x_i\right\|_{L^q(\mathcal{M})} \simeq_q max\left\{\|(x_i)\|_{L^q(\mathcal{M};l_c^2)}, \|(x_i)\|_{L^q(\mathcal{M};l_r^2)}\right\},$$

For any finite sequence (x_i) *in* $L^q(\mathcal{M})$. On the other hand, if $1 \leq q < 2$ then

$$E\|r_i x_i\|_{L^q(\mathcal{M})} \simeq_q inf\left\{\|(y_i)\|_{L^q(\mathcal{M};l_c^2)} + \|(z_i)\|_{L^q(\mathcal{M};l_r^2)}\right\}.$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ in $L^q(\mathcal{M})$.

3.2. Lemma. Suppose that $0 < \alpha \leq \infty$. Let E be a symmetric quasi-Banach function space on $(0,\alpha)$. For any $q_E(0,\infty)$ define $\Phi_q : (0,1) \to (0,\infty)$ by $\Phi_q(t) = t^{-\frac{1}{q}}$. If $q_E < q$, then there is a constant $c_{q,E} > 0$ such that for any $f \in E$ we have

$$\left\|f \otimes \Phi_q\right\|_{E(0,\alpha)\times(0,1)} \leq c_{q,E}\|f\|_{E(0,\alpha)}. \quad (9)$$

Conversely if (9) holds for every $f \in E$ *then* $q_E \leq q$.

Proof. Let $q_E < q$ *and* $f \in E_+$. We can note first that

$$\left\|f \otimes \Phi_q\right\|_{E(0,\alpha)\times(0,1)} = \left\|f(s)t^{-\frac{1}{q}}\right\|_{E(0,\alpha)\times(0,1)} \leq$$

$$\left\|f(s)\sum_{n=0}^{\infty} 2^{\frac{n+1}{q}} \chi_{(2^{-n-1},2^{-n}]}\right\|_{E(0,\alpha)\times(0,1)} \leq$$

$$c\left(\sum_{n=0}^{\infty} 2^{\frac{r(n+1)}{q}} \left\|f(s)\chi_{(2^{-n-1},2^{-n}]}\right\|_{E(0,\alpha)\times(0,1)}\right)^{\frac{1}{r}};$$

Where $c > 0$ *and* $0 < r \leq 1$ as in (6).

Fix $q > q_0 > q_E$. Observe that $f(s)\chi_{(2^{-n-1},2^{-n}]}(t)$ has the same distribution on $(0,\alpha) \times (0,1)$ *as* $D_{2^{n+1}}f$ *on* $(0,\alpha)$. Hence, as E is symmetric, we finally obtain

$$\left\|f \otimes \Phi_q\right\|_{E(0,\alpha)\times(0,1)} \leq c\left(\sum_{n=1}^{\infty} 2^{r\left(\frac{n+`1}{q}\right)} \|D_{2^{n+1}}f(t)\|_{E(0,\alpha)}^r\right)^{\frac{1}{r}}$$

$$\leq cC_{q_0}\left(\sum_{n=1}^{\infty} 2^{\frac{r(n+1)}{q}} 2^{\frac{-r(n+1)}{q_0}}\right)^{\frac{1}{r}} \|f\|_{E(0,\alpha)} \lesssim_{q,E} \|f\|_{E(0,\alpha)} \text{ as } q > q_0.$$

To prove the second assertion notice first that since $\mu(D_s,f) \le D_s\mu(f)$ for all $s \in (0,\infty)$ and $f \in E$, it suffices to show that for all $s > 1$ and $f \in E_+$ we have $\|D_s f\|_E \le cs^{-\frac{1}{q}}\|f\|_E$. Fix $a \in (0,1)$ and observe that

$$\|f\otimes\Phi_q\|_{E(0,\alpha)\times(0,1)} = \left\|f(s)t^{-\frac{1}{q}}\right\|_{E(0,\alpha)\times(0,1)} \ge \left\|f(s)a^{-\frac{1}{q}}\chi_{\left(\frac{2}{a},a\right]}(t)\right\|_{E(0,\alpha)\times(0,1)} =$$

$$a^{-\frac{1}{q}}\left\|D_{\frac{2}{a}}f\right\|_{E(0,\alpha)},$$

where in the final step we use that $f(s)\chi_{\left(\frac{a}{2},a\right]}(t)$ has the same distribution on $(0,\alpha)$. Hence

$$\left\|D_{\frac{2}{a}}f\right\|_E \le a^{\frac{1}{q}}\|f\otimes\Phi_q\|_E \le c_{q,E}\left(\frac{2}{a}\right)^{-\frac{1}{q}2^{\frac{1}{q}}}\|f\|_E\,.$$

In other wise , for any $s \ge 2$ we obtain

$\|D_s f\|_E \le c_{q,E}2^{\frac{1}{q}}s^{-\frac{1}{q}}\|f\|_E$. Clearly this implies that $q_E \le q$.

3.3. Lemma. Let $0 < \alpha \le \infty. For\ 0 < q < \infty\ let\ \Phi_q$ in $(0,\alpha) \to (0,1)$ be given by $\Phi_q(t) = t^{-\frac{1}{q}}$. Let $f: (0,\alpha) \to [0,\infty]$ be measurable and a. e. finite. Then for every ≥ 0 .

$$d(v; f\otimes\Phi_q) = \int_{\{f\le v\}}\left(\frac{f(s)}{v}\right)^q ds + d(v;f).$$

Proof. By a change of variable.

$$\lambda\Big((s,t) \in (0,\alpha) \to (0,1): f(s)\Phi_q(t) > v\Big) = \int_0^1 \lambda\Big(s \in (0,\alpha): f(s)t^{-\frac{1}{q}} > v\Big)dt$$

$$= \int_0^1 \lambda(s \in (0,\alpha): f(s) > vu)qu^{q-1}du$$

$$= \int_0^\infty \lambda\left(s \in (0,\alpha): min\left(\frac{f(s)}{v},1\right) > u\right)qu^{q-1}du = \left\|min\left(\frac{f}{v},1\right)\right\|_{L^q(0,\alpha)}^q$$

$$= \int_{\{f\le v\}}\left(\frac{f(s)}{v}\right)^q ds + \lambda(s \in (0,\alpha): f(s) > v).$$

3.4.Lemma.(Chebyshev's inequality) Let $0 < q < \infty$ and $x \in L^q(\mathrm{M})$. Then for any > 0, $d(v;x) \le \frac{\|x\|_{L^q(\mathrm{M})}^q}{v^q}$.

3.5. Lemma. Let $\mathcal{M}$ be a semi finite von Neumann algebra with a normal , semi-finite , faithful trace τ satisfying $\tau(1) = \alpha$. Suppose E is a symmetric quasi-Banach function space on $(0,\alpha)$ which is p-convex for some $0 < p < \infty$ and suppose that for any finite sequence (x_k) of self-adjoint elements in $E(\mathcal{M})$ we have

$$\|\textstyle\sum_k r_k\otimes x_k\|_{E(L^\infty\bar{\otimes}\mathcal{M})} \lesssim_E \left\|(\sum_k x_k^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}.$$

Then, for any finite sequence (x_k) in $E(\mathcal{M})$,

$$\|\textstyle\sum_k r_k\otimes x_k\|_{E(L^\infty\bar{\otimes}\mathcal{M})} \lesssim_E max\left\{\|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)}\right\}.$$

On the other hand, if we have

$$\mathbb{E}\|\textstyle\sum_k r_k x_k\|_{E(\mathcal{M})} \lesssim_E \|\sum_K x_k^2\|_{E(\mathcal{M})}$$

for any finite sequence (x_k) of self-adjoint elements in $E(\mathcal{M})$, then for any finite sequence (x_k) in $E(\mathcal{M})$,

$$\mathbb{E}\|\textstyle\sum_k r_k x_k\|_{E(\mathcal{M})} \lesssim_E max\left\{\|(x_k)\|_{E(\mathcal{M};l_c^2)}, \|(x_k)\|_{E(\mathcal{M};l_r^2)}\right\}.$$

Proof. Let $(x_k)_{k=1}^n$ be any finite sequence in $E(\mathcal{M})$, put

$x_k = y_k + iz_k$, $y_k^* = y_k$, $z_k^* = z_k$, and notice that

$0 \le y_k^2, z_k^2 \le y_k^2 + z_k^2 = \frac{1}{2}(x_k^*x_k + x_kx_k^*), 1 \le k \le n$. Hence,

$(\sum_k y_k^2)^{\frac{1}{2}}, (\sum_k z_k^2)^{\frac{1}{2}} \leq \left(\sum_k \frac{1}{2}(|x_k|^2 + |x_k^*|^2)\right)^{\frac{1}{2}} = \frac{1}{\sqrt{x}}(\sum_k(|x_k|^2 + |x_k^*|^2))^{\frac{1}{2}}$. The assertion now readily follow from a straightforward computation.

3.6. Theorem. Let. $0 < \alpha \leq \infty$ and let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, faithful, semi-finite trace τ satisfying $\tau(1) = \alpha$. Suppose E is a symmetric quasi-Banach function space on $(0, \alpha)$ whch is p-convex for some $0 < p < \infty$ *and satisfies* $q_E < \infty$. Then

$$\|r_i \otimes x_i\|_{E(L^\infty \bar{\otimes} \mathcal{M})} \lesssim_E max\left\{\|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)}\right\}, \tag{10}$$

For any Rademacher sequence (r_i) and any finite sequence (x_i) in $E(\mathcal{M})$.

Proof. By Lemma 3.5, it suffices to consider the case where $x_1, \ldots, x_n$are self-adjoint. We begin by showing that for any $q \in [1, \infty)$ *and* $v > 0$

$$d(v; \textstyle\sum_i r_i \otimes x_i) \leq C_q d(v; f \otimes \Phi_q),$$

where $f: (0, \alpha) \to [0, \infty]$ *and* $\Phi_q: (0,1) \to (0, \infty)$ are defined by $f(s) = \mu_s\left((\sum_i x_i^2)^{\frac{1}{2}}\right)$ *and* $d_q(t) = t^{-\frac{1}{q}}$ *and* C_q is constant depending only on q. Fix $v > 0$. Define $\hat{e}_v = 1 \otimes e_v$, where $e_v = e^{(\sum_i x_i^2)^{\frac{1}{2}}}[0, v]$, then $\hat{e}_v \frac{1}{v} = 1 \otimes e \frac{1}{v} = 1 \otimes e^{(\sum_i x_i^2)^{\frac{1}{2}}}(v, \infty)$. Since $d(v; a + b) \leq d\left(\frac{v}{2}; q\right) + d\left(\frac{v}{4}; \hat{e}\, \frac{1}{v} \sum_i r_i \otimes x_i \hat{e}\, \frac{1}{v}\right)$. Recall that if $y \in S(\tau)$ *and* e is a finite trace projection in , then $\mu_t(ye) = \mu_t(ey) = 0$ *for* $t > \tau(e)$. Hence

$$d\left(v; \hat{e}\, \frac{1}{v} \textstyle\sum_i r_i \otimes x_i \hat{e}_v\right) \leq \mathbb{E} \otimes \tau\left(\hat{e}\, \frac{1}{v}\right) = \tau\left(e\, \frac{1}{v}\right) = d\left(v; (\sum_i |x_i|^2)^{\frac{1}{2}}\right) = d(v; f),$$

and analogously,

$$d\left(v; \hat{e}_v \textstyle\sum_i r_i \otimes x_i \hat{e}\, \frac{1}{v}\right), d\left(v; \hat{e}\, \frac{1}{v} \sum_i r_i \otimes x_i \hat{e}\, \frac{1}{v}\right) \leq d\left(v; (\sum_i |x_i|^2)^{\frac{1}{2}}\right) = d(v; f).$$

We estimate the remaining term using the noncommutative Khintchine inequality in $L^q(M)$ (Theorem3.1) see for example (Lindenstrauss & Tzafriri, 1979), (Theorem 1.e.13) and Lemma 3.4. The proof is complete.

We can obtain the following result of Theorem 3.6 for spaces with $q_E < 2$.

3.7. Theorem. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful trace τ satisfying $\tau(1) = \alpha. E$ is a symmetric quasi-Banach function space on $(0, \alpha)$ which is p-convex for some $0 < p < \infty$ and suppose $q_E < 2$. Then for any finite sequence (x_i) *in* $E(\mathrm{M})$ we have

$$\|\textstyle\sum_i r_i \otimes x_i\|_{E(L^\infty \bar{\otimes} \mathcal{M})} \leq_{c_E} inf\left\{\|(y_i)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\right\}, \tag{11}$$

Where the infimum is taken over the decompositions $x_i = y_i + z_i$ in $E(\mathcal{M})$. If E is a symmetric Banach function space on $(0, \infty)$ which is separable or the dual of a separable space and satisfies $q_E < 2$ then

$$\|\textstyle\sum_i r_i \otimes x_i\|_{E(\mathrm{L}^\infty \bar{\otimes} \mathcal{M})} \simeq_E inf\left\{\|(y_i)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\right\}.$$

Proof. Fix y_i, z_i *in* $E(\mathcal{M})$ *such that* $x_i = y_i + z_i$ *for* $1 \leq i \leq n$. Fix $v > 0$ and $q_E < q < 2$. Define $y = (|y_i|^2)^{\frac{1}{2}}, z = (\sum |z_i^*|^2)^{\frac{1}{2}}$ and set $\hat{e}_v^y = 1 \otimes e_v^y, \hat{e}_v^z = 1 \otimes e_v^z$. Set $f_y(s) = \mu_s(y), f_z(s) = \mu_s(z)$ and $f(s) = \mu_s(y + z)$. We first note that

$$d(v; \textstyle\sum_i r_i \otimes x_i) \leq d\left(\frac{v}{16}; \hat{e}_v^y \hat{e}_v^z \sum_i r_i \otimes x_i \hat{e}_v^y \hat{e}_v^z\right) + d\left(\frac{v}{16}; \hat{e}_v^y \hat{e}_v^z \sum_i r_i \otimes x_i \hat{e}_v^y (\hat{e}_v^z)^\perp\right)$$
$$+ d\left(\frac{v}{8}; \hat{e}_v^y \hat{e}_v^z \textstyle\sum_i r_i \otimes x_i (\hat{e}_v^y)^\perp\right) + d\left(\frac{v}{4}; \hat{e}_v^y (\hat{e}_v^z)^\perp \sum_i r_i \otimes x_i\right) + d\left(\frac{v}{2}; (\hat{e}_v^y)^\perp \sum_i r_i \otimes x_i\right).$$

Reasoning as in the proof of Theorem 3.6 we obtain by Chebyshev's inequality , Kahane's inequality and the non-commutative Khintchine inequality for $L^q(\mathcal{M})$,

$$d\left(v; \hat{e}_v^y \hat{e}_v^z (\textstyle\sum_i r_i \otimes x_i) \hat{e}_v^y \hat{e}_v^z\right) \lesssim_q d\left(v; f_y \otimes \Phi_q\right) + d\left(v; f_z \otimes \Phi_q\right) \leq d\left(v; f \otimes \Phi_q\right).$$

Moreover,

$$d\left(\frac{v}{16}; \hat{e}_v^y \hat{e}_v^z \sum_i r_i \otimes x_i \hat{e}_v^y (\hat{e}_v^z)^\perp\right) \leq \mathbb{E}\otimes\tau((\hat{e}_v^z)^\perp) = d(v;z) \leq d(v; f_z \otimes \Phi_q),$$

and similarly we find that

$$d\left(\frac{v}{8}; \hat{e}_v^y \hat{e}_v^z \sum_i r_i \otimes x_i \hat{e}_v^y (\hat{e}_v^z)^\perp, d\left(\frac{v}{4}; \hat{e}_v^y (\hat{e}_v^z)^\perp \sum_i r_i \otimes x_i\right), d\left(\frac{v}{2}; (\hat{e}_v^y)^\perp \sum_i r_i \otimes x_i\right)\right)$$

are bounded by $d(v; f\otimes\Phi_q)$. We conclude that there is a constant C_q depending only on q such that for all > 0 ,

$$d(v; \textstyle\sum_i r_i \otimes x_i) \leq C_q d(v; f\otimes\Phi_q).$$

Since the dilation $D_{C_q^{-1}}$ is bounded on E, we obtain by Lemma 3.3

$$\|\textstyle\sum_i r_i \otimes x_i\|_{E(L^\infty \bar{\otimes} M)} \lesssim_{q,E} \|f\otimes\Phi_q\|_{E(0,\alpha)\times(0,1)} \lesssim_{q,E} \|f\|_{E(0,\alpha)}$$

$$\lesssim_{q,E} \left\|(\textstyle\sum_i |y_i|^2)^{\frac{1}{2}}\right\|_{E(M)} + \left\|(\textstyle\sum_i |z_i^*|^2)^{\frac{1}{2}}\right\|_{E(M)}.$$

By taking the infimum over all possible decompositions $x_i = y_i + z_i$ in $E(M)$ we obtain (11). The final statement follows from (Lemardy & Sukochev, 2008), Theorem 1.1.(1), which states that the reverse of the inequality in (11) holds if E is separable or dual of a separable space and $q_E < \infty$.

3.8. Corollary. Let $\mathcal{M}$ be semi-finite von Neumann algebra . Suppose E is a separable symmetric Banach function space on $(0,\infty)$ with $p_E > 1$. Then for any finite sequence $(x_i)\ in\ E(\mathcal{M})$,

$$inf\left\{\|(y_i)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\right\} \lesssim_E \|\textstyle\sum_i r_i \otimes x_i\|_{E(L^\infty \bar{\otimes} \mathcal{M})}, \tag{12}$$

where the infimum is taken over all decompositions $x_i = y_i + z_i$ in $E(\mathcal{M})$. If $p_E > 2$ then

$$max\left\{\|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)}\right\} \lesssim_E \|\textstyle\sum_i r_i \otimes x_i\|_{E(L^\infty \bar{\otimes} \mathcal{M})}.$$

In the proof of theorem 3.6 and 3.7 we can use the non-commutative Khintchine inequalities in (Junge & Xu, 2008), remark 3-5 to obtain the following version where the Rademacher sequence is replaced by a sequence of independent non-commutative variables.

3.9. Corollary. Let $\mathcal{M}, N$ be a von Neumann algebras equipped with normal, faithful finite trace $and\ \sigma$, respectively, satisfying $\tau(1) = \alpha$ and $\sigma(1) = \beta$. Suppose E is a p-convex $0 < p < \infty$ symmetric quasi-Banach function space on $(0, \alpha\beta)\ with\ q_E < \infty$. Let $q > max\{2, q_E\}$ and $(\alpha_i)_{i\geq 1}$ be a sequence in $L^q(N)$ which is independent with respect to σ, satisfying $\sigma(\alpha_i) = 0$ and is such that $d_q = sup_{i\geq 1}\|\alpha_i\|_q < \infty$. Then

$$\|\textstyle\sum_i \alpha_i \otimes x_i\|_{E(N\bar{\bar{\otimes}}\mathcal{M})} \lesssim_{E,d_q} max\left\{\|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)}\right\},$$

for any finite sequence $(x_i)\ in\ E(M)$. If $q_E < 2$ then

$$\|\textstyle\sum_i \alpha_i \otimes x_i\|_{E\ (N\bar{\otimes}\mathcal{M})} \lesssim_{E,d_q} inf\left\{\|(y_k)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\right\},$$

Where the infimum is taken over all decompositions $x_i = y_i + z_i\ in\ E(\mathcal{M})$.

3.10. Theorem. Suppose E is a symmetric quasi-Banach function on $(0,\infty)$ which is p-convex for some $0 < p < \infty$. Then the following are equivalent.

(i) The inequality (10) holds for any semi-finite von Neumann algebra $\mathcal{M}$;

(ii) $q_E < \infty$.

Proof. Moreover, if this is the case and if E is either a separable symmetric Banach function space or the dual of separable symmetric space, then

$$\|(x_k)\|_{E(\mathcal{M};l_c^2)+E(\mathcal{M};l_r^2)} \lesssim_E \|\textstyle\sum_k r_k \otimes x_k\|_{E(L^\infty \bar{\otimes} \mathcal{M})} \lesssim_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)}.$$

It remains to prove $(i) \Rightarrow (ii)$. Suppose $q_E = \infty$. It follows by (Lindenstrauss & Tzafriri, 1979), proposition 2.b.7, that for every $\varepsilon > 0$ there exists a sequence $(x_i)_{i=1}^n$ of mutually disjoint independent distributed in E such that $\|x_i\| = 1$ for all i and $1 \leq \|\sum_{i=1}^n x_i\|_{E(0,\alpha)} < 1 + \varepsilon$. One can show that (10) cannot hold for $M = L^\infty(0,1)$, by proceeding as in the proof of

(Junge, 2002), corollary 1. The final assertion follows by (Lemardy & Sukochev, 2008), theorem 1.1.(1).

3.11. Theorem. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite faithful trace τ satisfying $\tau(1) = 1$. Suppose E is a symmetric quasi-Banach function space on $(0,\infty)$ which is p-convex for some $0 < p < \infty$ and $r - concave$ for some $< \infty$. Then

$$\mathbb{E}\|\sum_i r_i x_i\|_{E(\mathcal{M})} \lesssim_E max\left\{\|(x_i)\|_{E(\mathcal{M};l_c^2)}, \|(x_i)\|_{E(\mathcal{M};l_r^2)}\right\}, \tag{13}$$

for any Rademacher sequence (r_i) and any finite sequence (x_i) in $E(\mathcal{M})$.

Proof . By Lemma 3.5, it suffices to consider the case where $x_1, \dots, x_n$ are self-adjoin. Fix $q \geq 1$ such that $q > r$ and define $f:(0,\alpha) \to [0,\infty]$ by $f(s) = \mu_s\left((\sum_i x_i^2)^{\frac{1}{2}}\right)$ *and* $\Phi_q = t^{\frac{1}{q}}$. Since E is $q - concave$ for any $q \geq r$

$$\mathbb{E}\|\sum_i r_i x_i\|_{E(M)} \leq \left(\|\mu(\sum_i r_i x_i)\|_{E(M)}^q\right)^{\frac{1}{q}} \leq C_q(E)\left\|(\mathbb{E}\mu(\sum_i r_i x_i)^q)^{\frac{1}{q}}\right\|_E,$$

Where $C_q\ (E)$ is $q - concavity$ constant of E. Fix $v > 0$ and set $e_v = e^{(\sum_i |x_i|^2)^{\frac{1}{2}}}[0,v]$. Recall that $\mu_t(a+b) \leq \mu_{\frac{t}{2}}(a) + \mu_{\frac{t}{2}}(b)$ and $d(v;a+b) \leq d\left(\frac{v}{2};a\right) + d\left(\frac{v}{2};b\right)$ for all $a,b \in S(t)$. Hence, by the triangle inequality in $L^q(\Omega)$, we have for any $v > 0$

$$d\left(v;\left(\mathbb{E}(|\mu(\sum_i r_i x_i)|^q)^{\frac{1}{q}}\right)\right) \tag{14}$$

$$\leq d\left(\frac{v}{4};\left(\mathbb{E}\left|D_{\frac{1}{4}}\mu(e^v \sum_i r_i x_i e_v)\right|^q\right)^{\frac{1}{q}}\right) + d\left(\frac{v}{4};\left(\mathbb{E}\left|D_{\frac{1}{4}}\mu(e_v^{\perp} \sum_i r_i x_i e_v)\right|^q\right)^{\frac{1}{q}}\right).$$

$$+d\left(\frac{v}{4};\left(\mathbb{E}\left|D_{\frac{1}{4}}\mu(e_v \sum_i r_i x_i e_v^{\perp})\right|^q\right)^{\frac{1}{q}}\right) + d\left(\frac{v}{4};\left(\mathbb{E}\left|D_{\frac{1}{4}}\mu(e_v^{\perp} \sum_i r_i x_i e_v^{\perp})\right|^q\right)^{\frac{1}{q}}\right).$$

Recall that if e is a finite trace projection we have $\mu_t(ye) = \mu_t(ey) = 0$ for all $t \geq \tau(e)$. Therefore,

$$d\left(\frac{v}{4};\left(\mathbb{E}\left|D_{\frac{1}{4}}\mu(e_v^{\perp} \sum_i r_i x_i e_v)\right|^q\right)^{\frac{1}{q}}\right) \leq 4d\left(v;(\sum_i |x_i|^2)^{\frac{1}{2}}\right) = 4d(v;f),$$

and analogously,

$$d\left(\frac{v}{4};\left(\mathbb{E}\left|D_{\frac{1}{4}}\mu(e_v \sum_i r_i x_i e_v^{\perp})\right|^q\right)^{\frac{1}{q}}\right), d\left(\left(\frac{v}{4};\mathbb{E}\left|D_{\frac{1}{4}}\mu(e_v^{\perp} \sum_i r_i x_i e_v^{\perp})\right|^q\right)^{\frac{1}{4}}\right) \leq 4d(v;f) .$$

We estimate the final term in (12) using the non-commutative Khintchine inequality in $L^q(\mathcal{M})$ (Theorem 3.1) and Chebyshev's inequality (Lemma 3.5) . We obtain

$$v^q d\left(v;\left(\mathbb{E}\left(D_{\frac{1}{4}}\mu \sum_i r_i e_v x_i e_v\right)^q\right)^{\frac{1}{q}}\right) = v^q \lambda\left(\left(t \in (0,\infty): \mathbb{E}\left(\mu_{\frac{1}{4}} \sum_i r_i e_v x_i e_v\right)^q\right) > v^q\right)$$

$$\leq \int_0^\infty \mathbb{E}\left(\mu_{\frac{1}{4}}(\sum_i r_i e_v x_i e_v)^q\right) dt = \mathbb{E}\left\|D_{\frac{1}{4}}\mu(\sum_i r_i e_v x_i e_v)\right\|_{L^q(0,\infty)}^q$$

$$= 4^q \mathbb{E}\|\sum_i r_i e_v x_i e_v\|_{L^q(\mathrm{M})}^q \leq 4^q K_{q,1}^q \left(\mathbb{E}\|\sum_i r_i e_v x_i e_v\|_{L^q(\mathrm{M})}\right)^q$$

$$\leq 4^q K_{q,1}^q B_q^q \left\|(\sum_i |e_v x_i e_v|^2)^{\frac{1}{2}}\right\|_{L^q(\mathrm{M})}^q \leq 4^q K_{q,1}^q B_q^q \int_{\{f \leq v\}} f(s)^q ds,$$

where the last inequality follows by (10) and B_q *and* $K_{q,1}$ are the constants in the commutative Khintchine inequality and Kahane's inequality (Lindenstrauss & Tzafriri, 1979; theorem 1.e.13). By Lemma 3.4 we have $v^{-q}\int_{\{f \leq v\}} f(s)^q ds + d(v;f) = d(v; f \otimes \Phi_q)$ for all $v > 0$ and hence there is a constant $C_q > 0$ such that for any > 0 ,

$$d\left(v;(\mathbb{E}|\mu_t(\sum_i r_i x_i)|^q)^{\frac{1}{q}}\right) \leq C_q d\left(\frac{v}{4}; f\otimes\Phi_q\right).$$

Since the dilation operator $D_{C_q^{-1}}$ is bounded on E we obtain

$$\left\|(\mathbb{E}|\mu_t(\sum_i r_i x_i)|^q)^{\frac{1}{q}}\right\|_E \lesssim_{q,E} \left\|f\otimes\Phi_q\right\|_E .$$

Since the r-concavity of E implies that $q_E \leq r < q < \infty$, an application of Lemma 3.3 completes the proof.

3.12. Corollary. Let $\mathcal{M}$ be a semi-finite von Neumann algebra. Suppose E is a separable symmetric Banach function space on $(0,\infty)$ which is p-convex for some $p > 1$. Then, for any finite sequence (x_i) in $(\mathcal{M})$,

$$inf\left\{\|(y_i)\|_{E(\mathcal{M};l_c^2)} + \|(z_i)\|_{E(\mathcal{M};l_r^2)}\right\} \lesssim_E \mathbb{E}\|\sum_i r_i x_i\|_{E(\mathcal{M})}, \tag{15}$$

where the infimum is taken over all decompositions $x_i = y_i + z_i \; in \; E(\mathcal{M})$.

Now we obtain the following characterization of q-concave spaces.

3.13. Theorem. Let E be a separable quasi-Banach function space on $(0,\infty)$ which is p-convex for some $0 < p < \infty$. Then the following are equivalent.

(i) The inequality (13) holds for any semi-finite von Neumann algebra $\mathcal{M}$;

(ii) E is q-concave for some $q < \infty$.

Moreover, if his is the case and $p > 1$ we have

$\|(x_i)\|_{E(\mathcal{M};l_c^2)+E(\mathcal{M};l_r^2)} \lesssim_E \mathbb{E}\|r_i x_i\|_{E(\mathcal{M})} \lesssim_E \|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)}$,

For any finite sequence $(x_i) \; in \; E(\mathcal{M})$.

3.14. Corollary. Let E be a symmetric Banach function space on (o,α) and suppose E is 2-convex and q-concave for some $q < \infty$. Then , for any semi-finite von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(1) = \alpha$, any Rademacher sequence (r_i) and any finite sequence (x_i) in $E(\mathcal{M})$ we have

$\mathbb{E}\|\sum_i r_i x_i\|_{E(\mathcal{M})} \simeq_E \|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)} \simeq_E \|\sum_i r_i \otimes x_i\|_{E(L^\infty\bar{\otimes}\mathcal{M})}$. (16)

Proof. Since E is q-concave , it has order continuous norm and $q_E \leq q < \infty$. Hence, by theorem 3.6 and 3.11, it remains to show that

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)} \lesssim_E \mathbb{E}\|\sum_i r_i x_i\|_{E(\mathcal{M})}; \tag{17}$$

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)\cap E(\mathcal{M};l_r^2)} \lesssim_E \|\sum_i r_i \otimes x_i\|_{E(L^\infty\bar{\otimes}\mathcal{M})}. \tag{18}$$

To prove (17) recall that since E has Fatou norm and is 2-convex, $E(\mathcal{M})$ is 2-convex as well. Hence

$$\left\|(\sum_i|x_i|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})} = \left\|(\sum_i|r_i x_i|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})} = \left\|(\mathbb{E}|\sum_i r_i x_i|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}$$

$$\lesssim_E \left(\mathbb{E}\|\sum_i r_i x_i\|_{E(\mathcal{M})}^2\right)^{\frac{1}{2}} \lesssim \mathbb{E}\|\sum_i r_i x_i\|_{E(\mathcal{M})} ,$$

where in final inequality we apply Kahane,s inequality. By applying this to (x_i^*) we see that (17) holds.

Note that since $L^p\left(\Omega; L^p(\mathcal{M})\right) = L^p(L^\infty(\Omega)\bar{\otimes}\mathcal{M})$ holds isometrically for $2 \leq p < \infty$, the above shows that, for any finite sequence $(x_i)_{i=1}^n$ in $L^p(\mathcal{M})$,

$$\|(x_i)\|_{L^p(\mathcal{M};l_c^2)} \leq \|\sum_i r_i \otimes x_i\|_{L^p(L^\infty\bar{\otimes}\mathcal{M})}. \tag{19}$$

Since E is 2-convex and q-concave, E is an interpolation space for the couple (L^2, L^p) by Theorem 2.5. Hence, by the discussion following (7), $Rad_n(E)$ is a complemented subspace of $E(L^\infty\bar{\otimes}\mathcal{M})$ and by theorem 2.6, we obtain

$$\|(x_i)\|_{E(\mathcal{M};l_c^2)} \lesssim_E \|\sum_i r_i \otimes x_i\|_{E(L^\infty\bar{\otimes}\mathcal{M})}$$

By interpolation from (19). By applying this to $(x_i^*)_{i=1}^n$ we see that (18) holds.

3.15. Proposition. Let E be a fully symmetric quasi-Banach function space on $(0,\alpha)$ with Fatou quasi-norm and $1 < p_E \leq q_E < \infty$. For every $k \geq 1$,let $\varepsilon_k \in \mathcal{M}_{k-1}$ and suppose that $\|\xi_k\| \leq 1$ *and* ξ_k commutes with $\mathcal{M}_k$. Then , for any martingale difference sequence $(y_k)_{k=1}^{\infty}$ with respect to $(M_k)_{k=1}^{\infty}$ in $E(\mathcal{M})$ and any $n \geq 1$ we have

$\|\sum_{k=1}^{n} \xi_k y_k\|_{E(\mathcal{M})} \lesssim_E \|\sum_{k=1}^{n} y_k\|_{E(\mathcal{M})}$.

In particular, taking $\xi_k \in \{-1,1\}$ yields the well known fact that non-commutative martingale difference sequences are unconditional in $(\mathcal{M})$.

3.16. Lemma. Let E be a symmetric p-convex $(0 < p < \infty)$ quasi-Banach function space on $(0, \alpha)$ *with* $1 < p_E \leq q_E < \infty$ and suppose $\mathcal{M}$ is a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ satisfying $\tau(1) = \alpha$. Let $(\mathcal{M}_k)_{k=1}^{\infty}$ be an increasing sequence of von Neumann sub algebra such that $\tau|_{\mathcal{M}_k}$ is semi finite . Then we have the equivalences

$\mathbb{E}\|\sum_{k=1}^{n} r_k x_k\|_{E(\mathcal{M})} \simeq_E \|\sum_{k=1}^{n} x_k\|_{E(\mathcal{M})} \simeq_E \|\sum_{k=1}^{n} r_k \otimes x_k\|_{E(L^\infty \bar{\otimes} \mathcal{M})}$, (20)

For any Rademacher sequence (r_k) and any martingale difference sequence $(x_k)_{k=1}^{n}$

Proof. The first equivalence in (20) follows directly from the un conditionality of non-commutative martingale difference sequences in $E(\mathcal{M})$. For the second equivalence, observe that $(y_k) = (r_k \otimes x_k)$ is a martingale difference sequence with respect to the filtration $(L^\infty \bar{\otimes} \mathcal{M}_k)$. By applying proposition 3.15 with $\xi_k = r_k \otimes 1$

$\|\sum_{k=1}^{n} x_k\|_{E(\mathcal{M})} = \|\sum_{k=1}^{n} (r_k \otimes 1)(r_k \otimes x_k)\|_{E(L^\infty \bar{\otimes} \mathcal{M})} \lesssim_E \|\sum_{k=1}^{n} r_k \otimes x_k\|_{E(L^\infty \bar{\otimes} \mathcal{M})}$

The reverse inequality follows similarly from proposition 3.15 with $(y_k) = (1 \otimes x_k)$.

3.17. Proposition. Let E be a symmetric Banach function space on $(0, \infty)$ with $1 < p_E \leq q_E < \infty$ and suppose that E is either separable or is the dual of a separable symmetric space. Suppose $\mathcal{M}$ is a von Neumann algebra equipped with a normal semi finite , faithful trace τ, let $(\mathcal{M}_k)_{k=1}^{\infty}$, be an increasing sequence of von Neumann sub algebras such that $\tau|_{M_k}$ is semi-finite . Then for any finite martingale difference sequence (x_k) *in* $E(\mathcal{M})$ we have

$$\|(x_k)\|_{H_c^E + H_r^E} \lesssim_E \|\textstyle\sum_k x_k\|_{E(\mathcal{M})} \lesssim_E \|(x_k)\|_{H_c^E \cap H_r^E}.$$

Suppose that E is separable . If $p_E > 1$ and either $q_E < 2$ *or* E is 2-concave, then

$$\|\textstyle\sum_k x_k\|_{E(M)} \simeq_E \|(x_k)\|_{H_c^E + H_r^E} .$$

On the other hand, if either E is 2-convex and $q_E < \infty$ *or* $2 < p_E \leq q_E < \infty$ then

$$\|\textstyle\sum_k x_k\|_{E(M)} \simeq_E \|(x_k)\|_{H_c^E + H_r^E}.$$

3.18. Proposition. Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal semi-finite, faithful trace τ *and* $\widetilde{\mathcal{M}}$ a von Neumann sub algebra of $\mathcal{M}$ such that $\tau|_{\widetilde{M}}$ is again semi-finite . Let ξ be the conditional expectation with respect to $\widetilde{\mathcal{M}}$. Suppose E is a 2-convex symmetric Banach function space on $(0, \infty)$ with 2-convexity constant equal to 1 and suppose $E_{(2)}$ is fully symmetric. Then $\|.\|_{E(\mathcal{M};\xi)}$ define a norm on $E(\mathcal{M})$.

Proof. It clear that $\|.\|_{E(\mathcal{M};\xi)}$ is positive definite and homogeneous. It remains to show the triangle inequality. Let $x, y \in E(\mathcal{M})$ and fix $\alpha > 0$. Using that $|\alpha x - \alpha^{-1} y|^2 \geq 0$ it follows that

$|x + y|^2 \leq (1 + \alpha^2)|x|^2 + |(1 + \alpha^{-1})||y|^2$. Hence, as E is 2-convex with 2-convexity constant equal to 1,

$$\|\varepsilon|x + y|^2\|_{E_{(2)}(\mathcal{M})} \leq (1 + \alpha^2)\|\varepsilon|x|^2\|_{E_{(2)}(\mathcal{M})} + (1 + \alpha^{-2})\|\varepsilon|y|^2\|_{E_{(2)}(\mathcal{M})} .$$

Taking the infimum over all $\alpha > 0$ gives

$\|\varepsilon|x + y|^2\|_{E_{(2)}} \leq \left(\|\varepsilon|x|^2\|_{E_{(2)}}^{\frac{1}{2}} + \|\varepsilon|y|^2\|_{E_{(2)}}^{\frac{1}{2}} \right)^2$, which yields the result.

IV. Improved Non-Commutative Rosenthal's Inequality

We derive a generalization of Rosenthal's theorem to non-commutative symmetric spaces. Recall the following notion of conditional independence, which was introduced in

(Junge & Xu, 2008). Given a sequence (N_k) of conditional independence, which von Neumann algebra $\mathcal{M}$, we let $W^*((N_k)_k)$ denote the von Neumann subalgebra generated by (N_k) .

4.1. Definition. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite, faithful trace τ. Let (N_k) be a sequence of von Neumann sub algebra of $\mathcal{M}$ and N a common von Neumann subalgebra of the (N_k) such that $\tau|_N$ is semi-finite. We call (N_k) independent with respect to ξ_N for every k we have $\xi_N(xy) = \xi_N(x)\xi_N(y)$ for $x \in N_k$ and $y \in W^*\left((N_j)_{j\neq k}\right)$.

If a sequence (N_k) is independent with respect to ξ_N and $x_k \in N_k$ satisfy $\xi_N(a_k) = 0$, then (x_k) is a martingale difference sequence with respect to the filtration $\left(W^*(N_1, \ldots, N_k)\right)_{k=1}^{\infty}$. Also, if we let ξ_k denote the conditional expectation with respect to $W^*(N_1, \ldots, N_k)$, then by (5) one obtains $\xi_{k-1}(x_k) = \xi_N(x_k) = 0$.

4.2. Lemma. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with normal, semi-finite, faithful trace τ satisfying $\tau(1) = \alpha$ and let E be a p- convex $(0 < p < \infty)$ quasi-Banach function space on $(0, \alpha)$ which is an interpolation space for the couple (L^1, L^∞). Let (N_k) be a sequence of von Neumann subalgebra of the N_k such that $\tau|_N$ is semi-finite. Suppose (N_k) is independent with respect to $\mathcal{E}_N$. If $x_k \in E(N_k)$ satisfying $\mathcal{E}_N(x_k) = 0$, then for any Rademacher sequence (r_k) ,

$$\|\textstyle\sum_{k=1}^n x_k\|_{E(\mathcal{M})} \simeq_E E\|\sum_{k=1}^n r_k x_k\|_{E(N)} .$$

With constant depending only on E. If E is moreover q-concave for some $q < \infty$, then

$$\|\textstyle\sum_{k=1}^n x_k\|_{E(\mathcal{M})} \lesssim_E max\left\{\left\|(\sum_k |x_k|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}, \left\|(\sum_k |x_k^*|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}\right\}.$$

Proof. It suffices to show that for any sequence of signs $(\mathcal{E}_k)_{k=1}^n \subset \{-1,1\}^n$.

$\|\sum_{k=1}^n \mathcal{E}_k x_k\|_{E(\mathcal{M})} \lesssim_E \|\sum_{k=1}^n x_k\|_{E(\mathcal{M})}$. Define $N_+ = W^*(\{N_k : \mathcal{E}_k = 1\})$ and $N_- = W^*(\{N_k : \mathcal{E}_k = -1\})$. Note that if $\mathcal{E}_i = -1$, then by independent and (5) it readily follows that $\mathcal{E}_{N_+}(x_i) = \mathcal{E}_N(x_i) = 0$. Hence , $\mathcal{E}_{N_+}(\sum_{k=1}^n x_k) = \sum_{\mathcal{E}_{k=1}} x_k + \sum_{\mathcal{E}_k=-1} \mathcal{E}_{N_+}(x_k) = \sum_{\mathcal{E}_k=1} x_k$ and analogously , $\mathcal{E}_{N_-}(\sum_{k=1}^n x_k) = \sum_{\mathcal{E}_k=-1} x_k$. Since conditional expectations are bounded on $E(\mathcal{M})$ by a constant c_E depending only on E we obtain

$$\|\textstyle\sum_{k=1}^n \mathcal{E}_k x_k\|_{E(\mathcal{M})} = \left\|\sum_{\mathcal{E}_k=1} x_k - \sum_{\mathcal{E}_k=-1} x_k\right\|_{E(\mathcal{M})} = \left\|(\mathcal{E}_{N_+} - \mathcal{E}_{N_-})(\sum_{k=1}^n x_k)\right\|_{E(\mathcal{M})} \lesssim_E \|\sum_{k=1}^n x_k\|_{E(\mathcal{M})} .$$

The final statement follows from theorem 3.11.

4.3. Remark. Note that $2 < p_E \le q_E < \infty$ then E is an interpolation space for the couple (L^2, L^p), for any $p > q_E$. However, there are such spaces which are not q-concave for any $q < \infty$. Indeed, recall the Lorentz spaces $L^{p,q}$ on $(0, \infty)$ $(see$ [18], $section$ 4.4). Then the space $E = L^{3,\infty}$ has $p_E = q_E = 3$, but is $\infty - concave$.

4.4. Theorem. (non-commutative Rosenthal theorem). Let $\mathcal{M}$ be a semi-finite , faithful trace τ. Suppose E is a symmetric Banach function space on $(0, \infty)$ satisfying either of the following conditions

(i) E is an interpolation space for the couple (L^2, L^p) for some $2 \le p < \infty$ and E is q-concave for some $q < \infty$;

(ii) $2 < p_E \le q_E < \infty$.

Let (N_k) be a sequence of von Neumann subalgebra of $\mathcal{M}$ and N a common von Neumann subglgebra of the (N_k) such that $\tau|_N$ is semi-finite . Suppose (N_k) is independent with respect to $\mathcal{E} := \mathcal{E}_N$. Let $\mathcal{E}(x_k) = 0$ for all k. Then, for any n,

$$\|\textstyle\sum_{k=1}^n x_k\|_{E(\mathcal{M})} \simeq_F max\left\{\|diag(x_k)_{k=1}^n\|_{E(\mathcal{M}_n(\mathcal{M}))}, \|(x_k)_{k=1}^n\|_{E(\mathcal{M},\mathcal{E};l_c^2)}, \|(x_k)_{k=1}^n\|_{E(\mathcal{M},\mathcal{E};l_r^2)}\right\}. \quad (21)$$

Proof. Assume that x_k are bounded. Note that $\mathcal{E}$ is bounded on $E_{(2)}(\mathcal{M})$ under both condition (i) and (ii) by proposition 2.6. We first prove that the maximum on the right hand side is an interpolation space for the couple (L^2, L^p) for some $p < \infty$ under both condition (i) and (ii) , it follows from the discussion following (7) that the n-th Rademacher subspace $Rad_n(E)$ is $c_E - complemented\ in\ E(L^\infty\overline{\otimes}\mathcal{M})$, for some constant $c_E > 0$ independent of n. Recall that $L^q(\mathcal{M})$ has cotype q see (Pisier & Xu, 2003) i.e.

$$\|diag(x_k)_{k=1}^n\|_{L^q(N_n(\mathcal{M}))} = \left(\sum_{k=1}^n \|x_k\|_{L^q(\mathcal{M})}^q\right)^{\frac{1}{q}} \le \|\sum_{k=1}^n r_k \otimes x_k\|_{L^q(L^\infty\overline{\otimes}\mathcal{M})} .$$

By interpolation of this estimate for $q = 2\ and\ q = p$ we obtain

$$\|diag(x_k)_{k=1}^n\|_{E(N_n(\mathcal{M}))} \lesssim_E \|\sum_{k=1}^n r_k \otimes x_k\|_{E(L^\infty\overline{\otimes}\mathcal{M})} ,$$

and by Lemma 3.16

$$\|\sum r_k \otimes x_k\|_{E(L^\infty\overline{\otimes}\mathcal{M})} \simeq_E \|\sum_k x_k\|_{E(\mathcal{M})} .$$

Since the (N_k) are independent and we have $\mathcal{E}(x_k) = 0$ for all k $\left(so\ \mathcal{E}(x_k^* x_j) = \mathcal{E}(x_k^*)\mathcal{E}(x_j) = 0\ if\ j \ne k\right)$ we have by boundedness of $\mathcal{E}$ in $E_{(2)}(\mathcal{M})$,

$$\left\|\sum_k \mathcal{E}(x_k^* x_k)^{\frac{1}{2}}\right\|_{E(\mathcal{M})} = \|\sum_K \mathcal{E}(x_k^* x_k)\|_{E_{(2)}(\mathcal{M})}^{\frac{1}{2}} =$$

$$\|\mathcal{E}(\sum_k x_k^*)(\sum_k x_k)\|_{E_{(2)}(\mathcal{M})}^{\frac{1}{2}} \lesssim_E \|\sum_k x_k\|_{E(\mathcal{M})} ,$$

and by applying this to the sequence (x_k^*) we get

$$\left\|(\sum_k \mathcal{E}(x_k x_k^*))^{\frac{1}{2}}\right\|_{E(\mathcal{M})} \lesssim_E \|\sum_k x_k\|_{E(\mathcal{M})}.$$

We now prove the reverse inequality in (21). By Lemma 4.2 (case (i)) or proposition 3.17 (case (ii)), respectively we have

$$\|\sum_k x_k\|_{E(\mathcal{M})} \lesssim_E max\left\{\left\|(\sum_k x_k^* x_k)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}, \left\|\sum_K (x_k x_k^*)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}\right\}. (22)$$

By the quasi-triangle inequality in $E_{(2)}(\mathcal{M})$ we have

$$\left\|(\sum_k x_k^* x_k)^{\frac{1}{2}}\right\|_{E(\mathcal{M})} \lesssim_E \left(\|\sum_k x_k^* x_k - \mathcal{E}(x_k^* x_k)\|_{E_{(2)}(\mathcal{M})} + \|\sum_k (x_k^* x_k)\|_{E_2(\mathcal{M})}\right)^{\frac{1}{2}} .$$

Notice that $\left(|x_k|^2 - \mathcal{E}(|x_k|^2)\right)_{k\ge1}$ is independent with respect to $\mathcal{E}$, self-adjoint and , moreover, $\mathcal{E}\left(|x_k|^2 - \mathcal{E}(|x_k|^2)\right) = 0\ for\ all\ k$. Hence it is a martingale difference sequence and we obtain again by Lemma 4.2 (case (i)) or proposition 3.17 case (ii), since in this case $(1 < p_{E_{(2)}}, q_{E_{(2)}} < \infty)$, respectively,

$$\|\sum_k x_k^* x_k - \mathcal{E}(x_k^* x_k)\|_{E_{(2)}((\mathcal{M}))} \lesssim_E \left\|\left(\sum_k \left(x_k^* x_k - \mathcal{E}(x_k^* x_k)\right)^2\right)^{\frac{1}{2}}\right\|_{E_{(2)}((\mathcal{M}))}$$

$$\lesssim_E \left\|(|x_k|^4)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M})} + \left\|\left(\sum_k\ \left(\mathcal{E}(|x_2|^2)\right)^2\right)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M})},$$

where in the final inequality we use the quasi-triangle inequality in $E_{(2)}(\mathcal{M}; l_c^2)$. Let $x = col(|x_k|)\ and\ y = diag(|x_k|). Since\ \mu(xy) \prec\prec \mu(x)\mu(y)$, we obtain

$$\left\|(\sum_k |x_k|^4)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M})} = \left\|(x^* y^* y x)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M}_n(\mathcal{M}))} = \|yx\|_{E_{(2)}(\mathcal{M}_n(\mathcal{M}))}$$

$$\lesssim_E \|\mu(x)\mu(y)\|_{E_{(2)}} = \left\|\mu(x)^{\frac{1}{2}}\mu(y)^{\frac{1}{2}}\right\|_E^2 \|y\|_{E(\mathcal{M}_n(\mathcal{M}))}\|x\|_{E(\mathcal{M}_n(\mathcal{M}))}$$

$$= \|diag(x_k)\|_{E(\mathcal{M}_n(\mathcal{M}))} \left\|(\sum_k |x_k|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})},$$

where in the final inequality we use the $H\ddot{o}lder - type\ inequality$ in (Lindenstrauss & Tzafriri, 1979), proposition 1.d.2 (i).

Let $\mathcal{E}_n$ be the conditional expectation in $E(\mathcal{M}_n(\mathcal{M}))$ with respect to the von Neumann subalgebra $M_n(N)$, i.e. $\mathcal{E}_n = \mathcal{E}\otimes 1_{\mathcal{M}_n(c)}$. Writing $z = col(|x_k|^2)$, we have $\mathcal{E}_n(z) = col(\mathcal{E}|x_k|^2)$ and so by boundedness of $\mathcal{E}_n$ in $E_{(2)}(\mathcal{M}_n(\mathcal{M}))$,

$$\left\|(\textstyle\sum_k(\mathcal{E}|x_k|^2)^2)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M})} = \left\|\left((\mathcal{E}_n(z))^*\mathcal{E}_n(z)\right)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M}_n(\mathcal{M}))}$$

$$= \|\mathcal{E}_n(z)\|_{E_{(2)}(\mathcal{M}_n(\mathcal{M}))} \lesssim_E \|z\|_{E_{(2)}(\mathcal{M}_n(\mathcal{M}))} = \left\|(\textstyle\sum_k|x_k|^4)^{\frac{1}{2}}\right\|_{E_{(2)}(\mathcal{M})}.$$

Putting the estimates together we arrive at

$$\left\|(\textstyle\sum_k|x_k|^2)^{\frac{1}{2}}\right\|_E \lesssim_E \left(\|diag(x_k)\|_{E(\mathcal{M}_n(\mathcal{M}))}\left\|(\textstyle\sum_k|x_k|^2)^{\frac{1}{2}}\right\|_E + \left\|(\textstyle\sum_k \mathcal{E}(|x_k|^2))^{\frac{1}{2}}\right\|_E^2\right)^{\frac{1}{2}}.$$

In the other words, if we set $= \left\|(\sum_k|x_k|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}$, $b = \|diag(x_k)\|_{E(\mathcal{M}_n(\mathcal{M}))}$ and $c = \left\|(\sum_k \mathcal{E}|x_k|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})}$, we have $a^2 \lesssim_E ab + c^2$. Solving this quadratic equation we obtain $a \lesssim_E max\{b, c\}$, or ,

$$\left\|(\textstyle\sum_k|x_k|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})} \lesssim_E max\left\{\|diag(x_k)\|_{E(\mathcal{M}_n(\mathcal{M}))}, \left\|(\textstyle\sum_k \mathcal{E}(|x_k|^2))^{\frac{1}{2}}\right\|_{E(\mathcal{M})}\right\}.$$

Applying this to the sequence (x_k^*) gives

$$\left\|(\textstyle\sum_k|x_k^*|^2)^{\frac{1}{2}}\right\|_{E(\mathcal{M})} \lesssim_E max\left\{\|diag(x_k)\|_{E(\mathcal{M}_n(\mathcal{M}))}, \left\|(\textstyle\sum_k \mathcal{E}(|x_k^*|^2))^{\frac{1}{2}}\right\|_{E(\mathcal{M})}\right\}.$$

The result follows by (22). The final assertion follows by a straightforward.

4.5. Remark.Theorem 4.4 generalizes the Rosenthal inequalities for commutative Banach function spaces (Takesaki, 1972; remark 7) and for non-commutative $L^p - spaces$ (Pisier & Xu, 2003; Theorem 2.1). These two results can be recovered by taking $\mathcal{M} = L^\infty(\Omega), N = C$ in the first case and by setting $E = L^p$ in the second. Note , however, that the proof in (Pisier & Xu, 2003) is also valid for Haagerup $L^p - spaces$ (i.e., if τ is not a trace).

4.6. Corollary. Let $q \le p < \infty$, $and\ q > 2. Let\ \mathcal{M}$ be a von Neumann algebra equipped with normal, faithful state Φ. Suppose (N_k) is a sequence of von Neumann sub algebras of $\mathcal{M}\ and\ N$ is a common von Neumann sub algebra of N_k is independent with respect to $\mathcal{E} = \mathcal{E}_N$. Let $x_k \in L^p(N_k)$ be such that $\hat{\mathcal{E}}(x_k) = 0$ for all k then ,

$$\|\textstyle\sum_k x_k\|_{L^p(\mathcal{M})} \simeq_p max\left\{\|(x_k)\|_{l^p(L^p(\mathcal{M}))}, \|(x_k)\|_{L^p(\mathcal{M},\hat{\mathcal{E}};l_c^q)}, \|(x_k)\|_{L^p(\mathcal{M},\hat{\mathcal{E}};l_r^q)}\right\}.$$

Proof. The case where $p = q$ is trivial, suppose $q < p < \infty$. In (Haagerup, Junge, & Xu, 2010), Lemma 6.14, it is shown that the von Neumann sub algebra $R(N_k)$ are independent with respect to $\mathcal{E}_{R(N)} = \hat{\mathcal{E}}_N$ whenever (N_k) is independent with respect to $\mathcal{E}_N$. By Theorem 4.4 we obtain the Rosenthal inequalities in $L^{p,\infty}(R(\mathcal{M}))$ for bounded elements. In (Junge & Xu, 2003), Lemma 1.1, it is shown that set $\mathcal{M}D^{\frac{1}{p}}$ dense in $L^p(\mathcal{M}, \Phi)$, so it suffices to show that the Rosenthal inequalities hold for the sequence $\left(x_k D^{\frac{1}{p}}\right)$, where $x_k \in N_k$. Set $e_m = e^D[0, m]$, then $x_k D^{\frac{1}{v}} e_m$ is bounded linear operator in $L^{p,\infty}(R(N_k), \tau)$. By the above we have

$$\left\|\textstyle\sum_{k=1}^n x_k D^{\frac{1}{v}} e_m\right\|_{p,\infty} \simeq_p \max\left\{\left\|\mathrm{diag}\left(x_k D^{\frac{1}{v}} e_m\right)\right\|_{L^{p,\infty}(\mathcal{M}_n(R(\mathcal{M})))}, \left\|\left(x_k D^{\frac{1}{v}} e_m\right)\right\|_{L^{p,\infty}(R(\mathcal{M}),\hat{\mathcal{E}};l_c^q)}, \left\|\left(x_k D^{\frac{1}{v}} e_m\right)\right\|_{L^{p,\infty}(R(\mathcal{M}),\hat{\mathcal{E}};l_r^q)}\right\}.$$

Since $L^{p,\infty}$ has Fatou norm, a standard argument shows that we can take the limit for $m \longrightarrow \infty$ to obtain the result.

References

Astashkin, S.V. (2010). Rademacher functions in symmetric spaces. *J. Mathematical Science, 169*(6), 725-886.

Astoshkin, S.V., & Maligranda, L. (2004). Interpolation between L_1 and L_p , $1 < p < \infty$. *Proc. Mer. Math. Soc., 132*(10), 2929-2938.

Bennett, C., & Sharpley, R. (1988). *Interpolation of operators*. Volume 129 of Pure and Applied Mathematics. Boston, MA: Academic Press Inc.

Haagerup, U., Junge, M., & Xu, Q. (2010). A reduction method for non-commutative L_p-spaces and applications. *Trans. Amer. Math. Soc., 362*(4), 2125-2165.

Johnson, W.B., & Schechtman, G. (1988). Martingal inequalities in rearrangement invariant function spaces. *Israel J. Math., 64*(3), 267-275.

Junge, M. (2002). Doob̓s inequality for non-commutative martingales. *J. Reine Angew. Math., 549*, 149-190.

Junge, M., & Xu, Q. (2003). Non-commutative Burkholder/ Rosenthal inequalities. *Ann. Probob., 31*(2), 948-995.

Junge, M., & Xu, Q. (2008). Non-commutative Burkholder/ Rosenthal inequalities. II: Applications. *Israel J. Math., 167*(1), 227-282.

Junge, M., & Zeng, Q. (2013). Non-commutative Bennet and Rosenthal inequalities. *Annals of Probability, 41*(6). 428-4316.

Kalton, N., & Montgonetry-Smith, S. (2003). *Interpolation of Banach spaces* (Vol. 2, pp. 1131-1175). Amsterdam: North-Holland.

Kreĭn, S.G., Petunin, Ju. I., & Semenov, E.M. (1982). *Interpolation of linear operators*. Volume 54 of Translations of Mathematical Mongraphs. Trans. from Russian by J. Szu’cs. American Mathematical Society, Providence, R.I.

Lemardy, C., & Sukochev, F.A. (2008). Bademacher averge on non-commutative symmetric spaces. *J. Func. Anal., 255*(1), 3329-3355.

Lindenstrauss, J., & Tzafriri, L. (1979). Classical Banach spaces. 11, Volume 97 of Ergebniss der Mathematit and threr Grenzgebiete [results in Mathematics and Related Areas]. Berlin: Springer-Verlag. Function spaces.

Lust-Piguard, F., & Pisier, G. (1991). Non-commutative Khintchine and Paley inequalities. *Aris. Mat., 29*(2), 241-260.

Montgmery-Smith, S. (n.d.). Personal communication.

Pisier, G., & Xu, Q. (2003). Non-commutative $L^p - spaces$. In: *Handbook of the geometry of Banach spaces* (Vol. 2, pp. 1459-1517). Amsterdam: North-Holland.

Rosenthal, H. P. (1970). On the subspaces of $L^p(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math., 8*, 273-303.

Takesaki, M. (1972). Conditional expectation in von Neumann algebra. *J. Functional Analysis, 9*, 306-321.

Takesaki, M. (1972). Conditional expectation in von Neumann algebras. *J. Functional Analysis, 9*, 306-321.

Toed Dirksen, S., De Pagter, B., Potapov, D., & Sukchev, F. (n.d.). Rosenthal inequalities in non-commutative symmetric spaces. Vici Subsidy 639, 033, 604 of the Netherland Organization for scientific Research (NOW).

Umegaki, H. (1954). Conditional expectation in an operator algebra. *Tôhoku Math. J., 2*(6), 177-181.