New Faisal Distribution with Properties and Applications

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Abstract. The lifetime data analysis is often used in applied sciences especially in physics, engineering and life insurance sector. In this respect the number of distributions have been developed. This study introduced a Faisal distribution and further expressed its mathematical and statistical properties including moment generating function, rht moment, mean deviations, Bonnferoni and Lorenz curve, reliability measures (survival function, hazard function, cumulative hazard function and reversed hazard function), order statistics, Reyni entropy. It also discussed maximum likelihood method of estimation and real life data applications of proposed distribution.

Key words: Quasi, Reliability, Reyni entropy, Order statistics

Introduction

In applied sciences generally engineering, Physics, demography and insurance sector, lifetime data analysis is crucial, under consideration the importance of lifetime data analysis probability distribution like exponential, Gamma, Wiebull, Lindley and Rayleigh distributions are utilizing. With the necessity of the time number of distributions taking part in lifetime data analysis like Sujatha, Akash, Shanker, Shambhu, Aradhana, Suja, Amerandra, Ishita, Odoma, Deyvia, Pranav and Rani distribution which are all of mixture formulation distribution. Lindley (1958) distribution is respectively given as:

$$f_L(\mathrm{y},\theta) = \frac{\theta^2}{\theta+1}(1+y)e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.1}$$

$$F_L(\mathrm{y},\theta) = 1-\frac{\theta y+\theta+1}{\theta+1}e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.2}$$

The Lindley probability density function is a mixture of $f_1(\mathrm{y})$ as an Exponential (θ) and $f_2(\mathrm{y})$ as a $Gamma(2,\theta) = \frac{\theta^2 y e^{-\theta y}}{\Gamma 2}$ with mixture proportions $P_1 = \frac{\theta}{\theta+1}$ and $P_2 = \frac{1}{\theta+1}$ is;

$$f_L(\mathrm{y},\theta) = P_1\, f_1(\mathrm{y}) + P_2 f_2(\mathrm{y}) \tag{1.3}$$

Shanker et al. (2013) introduced two parameter Lindley distribution which pdf and cdf defined as;

$$f_{TPLD}(\mathrm{y},\theta,\alpha) = \frac{\theta}{\alpha\theta+1}(\alpha+y)e^{-\theta y} \quad y>0;\ \theta>0;\ \alpha\theta>-1 \tag{1.4}$$

$$F_{TPLD}(\mathrm{y},\theta,\alpha) = 1-\left(1+\frac{\theta y}{\alpha\theta+1}\right)e^{-\theta y} \quad y>0;\ \theta>0;\alpha\theta>0 \tag{1.5}$$

Shanker (2015) developed a new one parameter Shanker distribution which pdf and cdf defined respectively following as;

$$f_{Sk}(\mathrm{y},\theta) = \frac{\theta^2(\theta+y)e^{-\theta y}}{\theta^2+1} \quad y>0;\ \theta>0 \tag{1.6}$$

$$F_{Sk}(\mathrm{y},\theta) = 1-\left\{1+\frac{\theta y}{\theta^2+1}\right\}e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.7}$$

Shanker (2016) Quasi Sujatha distribution pdf and cdf are defined as following:

$$f_{QS}(\mathrm{y},\theta,\alpha) = \frac{\theta^2}{\alpha\theta+\theta+2}(\alpha+\theta y+\theta y^2)e^{-\theta y} \quad y>0;\ \theta>0;\ \alpha>0 \tag{1.8}$$

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$$F_{QS}(\mathrm{y},\theta,\alpha) = 1-\left\{1+\left(\frac{\theta y+\theta+2}{\alpha\theta+\theta+2}\right)\right\}e^{-\theta y} \quad y>0;\ \theta>0;\ \alpha>0 \tag{1.9}$$

The Quasi Sujatha probability density function is a mixture of $f_1(\mathrm{y})$ as an Exponential (θ), $f_2(\mathrm{y})$ as a Gamma $(2,\theta)$ and $f_3(\mathrm{y})$ as a Gamma $(3,\theta)=\frac{\theta^3y^2e^{-\theta y}}{\Gamma 3}$ with mixture proportions, $P_1=\frac{\alpha\theta}{\alpha\theta+\theta+2}$ $P_2=\frac{\theta}{\alpha\theta+\theta+2}$ and $P_3=\frac{2}{\alpha\theta+\theta+2}$. Shanker (2017) developed Suja distribution pdf and cdf are;

$$f_{SQ}(\mathrm{y},\theta) = \frac{\theta^5}{\theta^4+24}(1+y^4)e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.10}$$

$$F_{QS}(\mathrm{y},\theta) = 1-\left\{1+\frac{24\theta y+12(\theta y)^2+4(\theta y)^3+(\theta y)^4}{\theta^4+24}\right\}e^{-\theta y} \quad y>0;\ \theta>0, \tag{1.11}$$

Shanker (2017) developed a Rama distribution pdf and cdf are;

$$f_{RM}(\mathrm{y},\theta) = \frac{\theta^4}{\theta^3+6}(1+y^3)e^{-\theta y} \quad y>0;\ \theta>0;\ \alpha>0 \tag{1.12}$$

$$F_{RM}(\mathrm{y},\theta) = 1-\left\{1+\frac{6\theta y+3(\theta y)^2+(\theta y)^3}{\theta^3+6}\right\}e^{-\theta y} \quad y>0;\ \theta>0,\alpha>0 \tag{1.13}$$

Shanker and Shukla (2017) introduced Ishtia distribution pdf and cdf are;

$$f_{Is}(\mathrm{y},\theta) = \frac{\theta^3(\theta+y^2)e^{-\theta y}}{\theta^3+2} \quad y>0;\ \theta>0 \tag{1.14}$$

$$F_{Is}(\mathrm{y},\theta) = 1-\left\{1+\frac{\theta y(\theta y+2)}{\theta^3+2}\right\}e^{-\theta y} \quad y>0;\ \theta>0 \tag{1.15}$$

Mussie and Shanker (2018) propose two-parameter Sujatha distribution (TPSD) with mixture proportions $P_1=\frac{\alpha\theta^2}{\alpha\theta^2+\theta+2}$ $P_2=\frac{\theta}{\alpha\theta^2+\theta+2}$ and $P_3=\frac{2}{\alpha\theta^2+\theta+2}$. Shukla (2018) developed a Pranav distribution pdf and cdf respectively are;

$$f_P(\mathrm{y},\theta) = \frac{\theta^4(\theta+y^3)e^{-\theta y}}{\theta^4+6} \quad y>0;\ \theta>0 \tag{1.16}$$

$$F_P(\mathrm{y},\theta) = 1-\left\{1+\frac{\theta y(\theta^2y^2+3\theta y+6)}{\theta^4+6}\right\}e^{-\theta y} \tag{1.17}$$

Shanker (2017) proposed Rani Distribution and discussed its properties and applications, pdf and cdf such that;

$$f_R(\mathrm{y},\theta) = \frac{\theta^5(\theta+y^4)e^{-\theta y}}{\theta^5+24} \quad y>0;\ \theta>0 \tag{1.18}$$

$$F_R(\mathrm{y},\theta) = 1-\left\{1+\frac{\theta y(\theta^3y^3+4\theta^2y^2+12\theta y+24)}{\theta^4+6}\right\}e^{-\theta y} \tag{1.19}$$

Shukla (2018) introduced a parameter name as Ram Awadh distribution pdf and cdf given as;

$$f_R(\mathrm{y},\theta) = \frac{\theta^6(\theta+y^5)e^{-\theta y}}{\theta^6+120} \quad y>0;\ \theta>0 \tag{1.20}$$

$$F_R(\mathrm{y},\theta) = 1-\left\{1+\frac{\theta y(\theta^4y^4+5\theta^3y^3+20\theta^2y^2+60\theta y+120)}{\theta^6+120}\right\}e^{-\theta y} \quad y>0;\ \theta>0, \tag{1.21}$$

Shanker and Shukla (2019) proposed a two-parameter lifetime distribution named Rama-Kamlesh distribution (RKD) pdf and cdf are;

$$f_{RK}(y;\ k,\theta) = \frac{\theta^{k+1}(1+y^k)e^{-\theta y}}{\theta^k+\Gamma(k+1)} \quad \theta>0; y>0;\ k>0;\ \alpha\geq 0 \tag{1.22}$$

$$F_{RK}(y;\ k,\theta) = 1-\frac{\{\theta^k(1+y^k)e^{-\theta y}+k\Gamma(k,\theta y)\}}{\{\theta^k+\Gamma(k+1)\}} \tag{1.23}$$

Faisal Distribution (FD)

In this work proposed a mixture distribution (MD) function with pdf and cdf defined as following respectively;

$$g_{\mathrm{MD}}(y;\ k,\theta,\alpha) = \frac{\theta^k(\theta\alpha^k+y^{k-1})e^{-\theta y}}{(\theta\alpha)^k+\Gamma k} \quad \theta>0; y>0;\ k>0,\alpha>-1 \tag{2.1}$$

$$G_{\mathrm{MD}}(y;\ k,\theta,\alpha) = 1-\frac{\{\Gamma(k,\theta y)+(\theta\alpha)^ke^{-\theta y}\}}{\{(\theta\alpha)^k+\Gamma k\}} \tag{2.2}$$

The mixture distribution function is mixture of proportions $P_1 = \frac{(\theta\alpha)^k}{(\theta\alpha)^k+\Gamma k}$ and $P_2 = \frac{\Gamma k}{(\theta\alpha)^k+\Gamma k}$ with $Exponential\ (\theta)$ and $Gamma\ (k,\theta) = \frac{\theta^k y^{k-1} e^{-\theta y}}{\Gamma k}$ distributions respectively. Where $\Gamma(k,\theta y)$ is an upper incomplete gamma function. The cumulative distribution function of the exponentiated family of distribution proposed by Mudholkar and Srivastava (1993) is defined as;

$$F(y;\ k,\theta,\alpha,\varphi) = \{G_{MD}(y;\ k,\theta,\alpha,)\}^{\varphi} \tag{2.3}$$

The exponentiated distributions proposed by Mudholkar and Srivastava (1993) due to the impact of an extra shape parameter $\varphi > 0$ are more flexible than the traditional models. Recently many exponentiated type distributions have been proposed in different studies which are listed by Ahmad et al. (2019).

To obtain new distribution function named "Faisal distribution" putting (2.2) into (2.3) which lead to new cdf and pdf of FD defined following respectively;

$$F_{FD}(y;\ k,\theta,\alpha,\varphi) = \left[1 - \frac{\{\Gamma(k,\theta y) + (\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k + \Gamma k\}}\right]^{\varphi} \tag{2.4}$$

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \varphi\left[1 - \frac{\{\Gamma(k,\theta y) + (\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k + \Gamma k\}}\right]^{\varphi-1} \frac{\theta^k(\theta\alpha^k + y^{k-1})e^{-\theta y}}{(\theta\alpha)^k + \Gamma k} \qquad \theta > 0; y > 0;\ k > 0;\ \alpha > -1;\ \varphi > 0 \tag{2.5}$$

Where $\gamma(k,\theta y)$ is incomplete lower function. If $\varphi = 1$ then the value of k=1 it reduced to exponential distribution, for k=2; $\alpha = 1$ is it is one parameter Shanker distribution, eq. (1.6) for $k = 3, \alpha = 1$ is Ishtia distribution eq. (1.14) for $k = 4;\ \alpha = 1$ it is Pranav distribution eq. (1.16), $k = 5;\ \alpha = 1$ is Rani distribution eq. (1.18), $k = 6;\ \alpha = 1$ is Ram Awadha distribution eq. (1.20), $\alpha = 0$ it reduced to $gamma\ (k,\theta)$.

Expansion of Faisal distribution (FD)

We know that expansion series;

$$(1-y)^n = \sum_{i=0}^{\infty}(-1)^i\binom{n}{i}y^n; \qquad (1+y)^n = \sum_{i=0}^{\infty}\binom{n}{i}y^n\ ;$$

From pdf of FD (2.5);

$$\left[1 - \frac{\{\Gamma(k,\theta y) + (\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k + \Gamma k\}}\right]^{\varphi-1} = \sum_{i=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma k\}^{i-j}(\theta\alpha)^{k(j-l)}e^{-(j-l)\theta y}\{\gamma(k,\theta y)\}^l}{\{(\theta\alpha)^k + \Gamma k\}^i}$$

Where $\gamma(k,\theta y) = \Gamma(k) - \Gamma(k,\theta y)$;

Put in (2.5) have;

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j}(\theta\alpha)^{k(j-l)}e^{-(j-l)\theta y}\{\gamma(k,\theta y)\}^l}{\{(\theta\alpha)^k + \Gamma(k)\}^i}\frac{\theta^k(\theta\alpha^k + y^{k-1})e^{-\theta y}}{(\theta\alpha)^k + \Gamma k} \tag{3.1}$$

$\gamma(k,y) = e^{-y}(y)^k\sum_{p=0}^{\infty}\frac{\Gamma k y^p}{\Gamma(k+p+1)}$ put in (3.1);

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\left[\begin{array}{c}\{\Gamma(k)\}^{i-j}(\theta\alpha)^{k(j-l)}e^{-(j-l+1)\theta y}\theta^k(\theta\alpha^k + y^{k-1})\\ \left\{e^{-\theta y}(\theta y)^k\sum_{p=0}^{\infty}\frac{\Gamma(k)(\theta y)^p}{\Gamma(k+p+1)}\right\}^l\end{array}\right]}{\{(\theta\alpha)^k + \Gamma(k)\}^{i+1}}$$

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \frac{\sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\left[\begin{array}{c}\{\Gamma(k)\}^{i-j}\theta^{k(j-l)}\alpha^{k(j-l)}e^{-(j-l+l+1)\theta y}\theta^{(lk)}(y)^{(lk)}\theta^k\\ (\theta\alpha^k + y^{k-1})\left\{\sum_{p=0}^{\infty}\frac{\Gamma(k)(\theta y)^p}{\Gamma(k+p+1)}\right\}^l\end{array}\right]}{\{(\theta\alpha)^k + \Gamma(k)\}^{i+1}}$$

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \frac{\sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\left[\begin{array}{c}\{\Gamma(k)\}^{i-j+l}\ \theta^{k(j-l)+lk+k}\ \alpha^{k(j-l)}e^{-(j+1)\theta y}(y)^{(lk)} \\ (\ \theta\alpha^k+y^{k-1})\left\{\sum_{p=0}^{\infty}\frac{(\theta y)^p}{\Gamma(k+p+1)}\right\}^l\end{array}\right]}{\{\ (\theta\alpha)^k+\Gamma(k)\}^{i+1}}$$

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \frac{\sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\left[\begin{array}{c}\{\Gamma(k)\}^{i-j+l}\ \theta^{k(j+1)}\ \alpha^{k(j-l)}e^{-(j+1)\theta y}y^{lk} \\ (\ \theta\alpha^k+y^{k-1})\left\{\sum_{p=0}^{\infty}\frac{(\theta y)^p}{\Gamma(k+p+1)}\right\}^l\end{array}\right]}{\{\ (\theta\alpha)^k+\Gamma(k)\}^{i+1}} \tag{3.2}$$

From Gradshetyn and Ryzhik (2000), series evaluation;

$$\left\{\sum_{p=0}^{\infty} a_p y^p\right\}^n = \sum_{p=0}^{\infty} D_p(\theta y)^p$$

$$D_0 = a_0^l;\ \ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(np-m+p)a_pD_{m-p}$$

Similarly;

$$\left\{\sum_{p=0}^{\infty}\frac{(\theta y)^p}{\Gamma(k+p+1)}\right\}^l = \sum_{p=0}^{\infty} D_p(\theta y)^p\ ;$$

$$a_p = \frac{1}{\Gamma(k+p+1)};\ D_0 = a_0^l;\ \ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p}$$

Put in (3.2)

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\ \theta^{k(j+1)}\ \alpha^{k(j-l)}e^{-(j+1)\theta y}y^{lk}(\ \theta\alpha^k+y^{k-1})\sum_{p=0}^{\infty}D_p(\theta y)^p}{\{\ (\theta\alpha)^k+\Gamma(k)\}^{i+1}}$$

Where;

$$a_p = \frac{1}{\Gamma(k+p+1)};\ D_0 = a_0^l;\ \ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p}$$

$$f_{FD}(y;\ k,\theta,\alpha,\varphi) = \sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\ \alpha^{k(j-l)}(\ \theta\alpha^k+y^{k-1})\sum_{p=0}^{\infty}D_p\ \theta^{k(j+1)+p}y^{lk+p}e^{-(j+1)\theta y}}{\{\ (\theta\alpha)^k+\Gamma(k)\}^{i+1}} \tag{3.3}$$

$$a_p = \frac{1}{\Gamma(k+p+1)};\ D_0 = a_0^l;\ \ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p}$$

Some Properties of Faisal Distribution

Moment generating function: The moment generating function of FD given as;

$$M_y(t) = \sum_{m=0}^{\infty}\sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\ \alpha^{k(j-l)}\ \theta^{k(j-l)-r+1}}{m!\{\ (\theta\alpha)^k+\Gamma\mathrm{k}\}^{i+1}(j+1)^{lk+p+r+k-1}}\Big[(j+1)^{k-1}\ (\theta\alpha)^k\sum_{p=0}^{\infty}D_p\Gamma(lk+p+r+1)+\sum_{p=0}^{\infty}D_p\Gamma(lk+p+k+r)\Big]\left(\frac{t}{\theta}\right)^m$$

$$\text{Where; } a_p = \frac{1}{\Gamma(k+p+1)};\ D_0 = a_0^l;\ \ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p} \tag{4.1}$$

Rth moments about origin: The rth moment about origin of FD defined as;

$$\mu_r{}' = \sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\ \alpha^{k(j-l)}\ \theta^{k(j-l)-r+1}}{\{\ (\theta\alpha)^k+\Gamma\mathrm{k}\}^{i+1}(j+1)^{lk+p+r+k-1}}\Big[(j+1)^{k-1}\ (\theta\alpha)^k\sum_{p=0}^{\infty}D_p\Gamma(lk+p+r+1)+\sum_{p=0}^{\infty}D_p\Gamma(lk+p+k+r)\Big]$$

$$\text{Where; } a_p = \frac{1}{\Gamma(k+p+1)};\ D_0 = a_0^l;\ \ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p}$$

$$r = 1,2,3,\ldots \tag{4.2}$$

1st four moment about origin are;

$$\mu_1{}' = \sum_{i,j,l=0}^{\infty}(-1)^l\binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\ \alpha^{k(j-l)}\ \theta^{k(j-l)}}{\{\ (\theta\alpha)^k+\Gamma\mathrm{k}\}^{i+1}(j+1)^{lk+p+k}}\Big[(j+1)^{k-1}\ (\theta\alpha)^k\sum_{p=0}^{\infty}D_p\Gamma(lk+p+2)+\sum_{p=0}^{\infty}D_p\Gamma(lk+p+k+1)\Big]$$

$\mu_2{}' = \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\, \alpha^{k(j-l)}\, \theta^{k(j-l)-1}}{\{(\theta\alpha)^k+\Gamma k\}^{i+1}(j+1)^{lk+p+k+1}}[(j+1)^{k-1}\,(\theta\alpha)^k \sum_{p=0}^{\infty} D_p\Gamma(lk+p+3) + \sum_{p=0}^{\infty} D_p\Gamma(lk+p+k+2)]$

$\mu_3{}' = \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\, \alpha^{k(j-l)}\, \theta^{k(j-l)-2}}{\{(\theta\alpha)^k+\Gamma k\}^{i+1}(j+1)^{lk+p+k+2}}[(j+1)^{k-1}\,(\theta\alpha)^k \sum_{p=0}^{\infty} D_p\Gamma(lk+p+4) + \sum_{p=0}^{\infty} D_p\Gamma(lk+p+k+3)]$

$\mu_4{}' = \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\, \alpha^{k(j-l)}\, \theta^{k(j-l)-3}}{\{(\theta\alpha)^k+\Gamma k\}^{i+1}(j+1)^{lk+p+k+3}}[(j+1)^{k-1}\,(\theta\alpha)^k \sum_{p=0}^{\infty} D_p\Gamma(lk+p+5) + \sum_{p=0}^{\infty} D_p\Gamma(lk+p+k+4)]$

Where; $a_p = \frac{1}{\Gamma(k+p+1)}$; $D_0 = a_0^l$; $D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p}$

Mean deviations

Generally totality of deviations usually either from the mean or the median is measured as the amount of variation in a population. These are known as the mean deviation about the mean and the mean deviation about the median and are defined respectively;

$$\varphi_1(y) = \int_0^{\infty} |Y-\mu| f_{FD}(\mathrm{y})dy \tag{4.3}$$

$\mu = E(y)$

$$\varphi_2(y) = \int_0^{\infty} |y-M| f_{FD}(\mathrm{y})dy \tag{4.4}$$

$M = Median(y)$

The measure of $\varphi_1(y)$ and $\varphi_2(y)$ can be calculated as:

$\varphi_1(y) = \int_0^{\mu}(\mu-y)f_{FD}(\mathrm{y})dy + \int_{\mu}^{\infty}(y-\mu)f_{FD}(\mathrm{y})dy$

$= \mu F(\mu) - \int_0^{\mu} yf_{FD}(\mathrm{y})dy - \mu[1-F(\mu)] + \int_{\mu}^{\infty} yf_{FD}(\mathrm{y})dy$

$= 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} yf_{FD}(\mathrm{y})dy$

$$= 2\mu F(\mu) + 2\int_0^{\mu} yf_{FD}(\mathrm{y})dy \tag{4.5}$$

$\varphi_2(y) = \int_0^{\infty}|y-M|f_{FD}(\mathrm{y})dy$

$\varphi_2(y) = \int_0^{M}(M-y)f_{FD}(\mathrm{y})dy + \int_{M}^{\infty}(y-M)f_{FD}(\mathrm{y})dy$

$= MF(M) - \int_0^{M} yf_{FD}(\mathrm{y})dy - M[1-F(M)] + \int_{M}^{\infty} yf_{FD}(\mathrm{y})dy$

$= 2MF(M) - \int_0^{M} yf_{FD}(\mathrm{y})dy - M + \int_{M}^{0} yf_{FD}(\mathrm{y})dy + \int_0^{\infty} yf_{FD}(\mathrm{y})dy$

$= -\int_0^{M} yf_{FD}(\mathrm{y})dy + \int_{M}^{0} yf_{FD}(\mathrm{y})dy + \mu$

$$= \mu - 2\int_0^{M} yf_{FD}(\mathrm{y})dy \tag{4.6}$$

So by using pdf (2.4) found that;

$$\int_0^{\mu} yf_{FD}(\mathrm{y})dy = \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\, \alpha^{k(j-l)}\, \theta^{k(j-l)}}{\{(\theta\alpha)^k+\Gamma k\}^{i+1}(j+1)^{lk+p+k}}[(j+1)^{k-1}\,(\theta\alpha)^k \sum_{p=0}^{\infty} D_p\gamma\{(lk+p+2),(j+1)\theta\mu\} + \sum_{p=0}^{\infty} D_p\gamma\{(lk+p+k+1),(j+1)\theta\mu\}] \tag{4.7}$$

where $\gamma(k,\theta\mu)$ *lower gamma incomplete function*

$$\int_0^{M} yf_{FD}(\mathrm{y})dy = \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\, \alpha^{k(j-l)}\, \theta^{k(j-l)}}{\{(\theta\alpha)^k+\Gamma k\}^{i+1}(j+1)^{lk+p+k}}[(j+1)^{k-1}\,(\theta\alpha)^k \sum_{p=0}^{\infty} D_p\gamma\{(lk+p+2),(j+1)\theta M\} + \sum_{p=0}^{\infty} D_p\gamma\{(lk+p+k+1),(j+1)\theta M\}] \tag{4.8}$$

$where\ \gamma(k+1,\theta M)\ lower\ gamma\ incomplete\ function$

Put (4.7) in (4.5) and (4.8) into (4.6) have following:

$$\varphi_1(y) = 2\Bigg[\mu \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}(\alpha)^{k(j-l)}(\theta)^{(jk+l)}e^{-j\theta y}(\mu)^{(k)(l+1)}\sum_{p=0}^{\infty}D_p(\theta\mu)^p}{\{(\theta\alpha)^k+\Gamma(k)\}^i} + \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\alpha^{k(j-l)}\theta^{k(j-l)}}{\{(\theta\alpha)^k+\Gamma k\}^{i+1}(j+1)^{lk+p+k}}\Big[(j+1)^{k-1}(\theta\alpha)^k\sum_{p=0}^{\infty}D_p\gamma\{(lk+p+2),(j+1)\theta\mu\} + \sum_{p=0}^{\infty}D_p\gamma\{(lk+p+k+1),(j+1)\theta\mu\}\Big]\Bigg] \quad (4.9)$$

$$\varphi_2(y) = \mu - 2\sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}(\alpha)^{k(j-l)}(\theta)^{k(j-l)-r+1}}{\{(\theta\alpha)^k+\Gamma(k)\}^{i+1}}V_M \quad (4.10)$$

$$V_M = \left[(\theta\alpha)^k\sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r}D_p\gamma\{(lk+p+r+1),(j+1)\theta M\} + \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r+k-1}D_p\gamma\{(lk+p+r+k),(j+1)\theta M\}\right]$$

Where

$a_p = \frac{1}{\Gamma(k+p+1)};\ D_0 = a_0^l;\ D_m = \frac{1}{ma_0}\sum_{p=1}^{m}(pl-m+p)a_pD_{m-p}$

$\{(lk+p+r+k),(j+1)\theta M\}$ Incomplete gamma function.

Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves by Bonferroni (1930) and Bonferroni and Gini indices have utilized in economics to study the variation in income, poverty, also in other fields like reliability, vital statistics, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q yf_{FD}(\mathrm{y})dy \quad (5.1)$$
$$= \frac{1}{p\mu}\int_0^\infty yf_{FD}(\mathrm{y})dy - \frac{1}{p\mu}\int_q^\infty yf_{FD}(\mathrm{y})dy$$
$$= \frac{1}{p} - \frac{1}{p\mu}\int_q^\infty yf_{FD}(\mathrm{y})dy$$

By using (2.5) pdf of FD find:

$$\int_0^q yf_{FD}(\mathrm{y})dy = \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}(\alpha)^{k(j-l)}(\theta)^{k(j-l)-r+1}}{\{(\theta\alpha)^k+\Gamma(k)\}^{i+1}}V_q \quad (5.2)$$

$$V_q = \left[(\theta\alpha)^k\sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r}D_p\gamma\{(lk+p+r+1),(j+1)\theta q\} + \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r+k-1}D_p\gamma\{(lk+p+r+k),(j+1)\theta q\}\right]$$

Put (5.2) into (5.1).

$$B(p) = \frac{1}{p\mu}\sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}(\alpha)^{k(j-l)}(\theta)^{k(j-l)-r+1}}{\{(\theta\alpha)^k+\Gamma(k)\}^{i+1}}V_q \quad (5.3)$$

$$V_q = \left[(\theta\alpha)^k\sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r}D_p\gamma\{(lk+p+r+1),(j+1)\theta q\} + \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r+k-1}D_p\gamma\{(lk+p+r+k),(j+1)\theta q\}\right]$$

$$L(p) = \frac{1}{\mu}\int_0^q yf_{FD}(\mathrm{y})dy$$
$$= \frac{1}{\mu}\int_0^\infty yf_{FD}(\mathrm{y})dy - \frac{1}{\mu}\int_q^\infty yf_{FD}(\mathrm{y})dy$$
$$= 1 - \frac{1}{\mu}\int_q^\infty yf_{FD}(\mathrm{y})dy \quad (5.4)$$

Put (5.2) into (5.4).

$$L(p) = 1 - \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\,(\alpha)^{k(j-l)}\,(\theta)^{k(j-l)-r+1}}{\mu\{\,(\theta\alpha)^k+\Gamma(k)\}^{i+1}}V_q \tag{5.5}$$

$$V_q = \left[\,(\theta\alpha)^k \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r} D_p\gamma\{(lk+p+r+1),(j+1)\theta q\} + \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r+k-1} D_p\gamma\{(lk+p+r+k),(j+1)\theta q\}\right]$$

Or both equivalent to $B(p) = \frac{1}{p\mu}\int_0^p F(y)^{-1}dy$ and $L(p) = \frac{1}{\mu}\int_0^p F(y)^{-1}dy$ where $q = F(p)^{-1}$

The Bonferroni indices is defined as:

$$B = 1 - \int_0^1 B(p)dp \tag{5.6}$$

Put (5.3) into (5.6) have,

$$B = 1 - \frac{1}{p\mu}\sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\,(\alpha)^{k(j-l)}\,(\theta)^{k(j-l)-r+1}}{\{\,(\theta\alpha)^k+\Gamma(k)\}^{i+1}}V_q \tag{5.7}$$

$$V_q = \left[\,(\theta\alpha)^k \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r} D_p\gamma\{(lk+p+r+1),(j+1)\theta q\} + \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r+k-1} D_p\gamma\{(lk+p+r+k),(j+1)\theta q\}\right]$$

The Gini indices defined as following:

$$G = 1 - 2\int_0^1 L(p)dp \tag{5.8}$$

Put (5.5) into (5.8) determined as:

$$G = 1 - 2\left[1 - \sum_{i,j,l=0}^{\infty}(-1)^l \binom{\varphi-1}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}\,(\alpha)^{k(j-l)}\,(\theta)^{k(j-l)-r+1}}{\mu\{\,(\theta\alpha)^k+\Gamma(k)\}^{i+1}}V_q\right] \tag{5.9}$$

$$V_q = \left[\,(\theta\alpha)^k \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r} D_p\gamma\{(lk+p+r+1),(j+1)\theta q\} + \sum_{p=0}^{\infty}\left(\frac{1}{j+1}\right)^{lk+p+r+k-1} D_p\gamma\{(lk+p+r+k),(j+1)\theta q\}\right]$$

Reliability Measures

There are different reliability measures namely Survival Function, Hazard Rate Function, Cumulative Hazard Function and Reversed Cumulative Hazard Function.

Let Y be a continuous random variable with pdf $f_{FD}(y;\ k,\theta,\alpha,\varphi)$ and cdf $F_{FD}(y;\ k,\theta,\alpha,\varphi)$ of FD. Then the Survival $S_{FD}(y;\ k,\theta,\alpha,\varphi)$, Hazard rate $h_{FD}(y;\ k,\theta,\alpha,\varphi)$, Cumulative hazard function $CH_{FD}(y;\ k,\theta,\alpha,\varphi)$ and $H_{FD}(y;\ k,\theta,\alpha,\varphi)$ Reversed cumulative hazard function are given below.

Survival function:

Let Y be a continuous random variable with pdf $f_{FD}(y;\ k,\theta,\alpha,\varphi)$ (2.5) and cdf $F_{FD}(y;\ k,\theta,\alpha,\varphi)$ (2.4) of FD the Survival function obtain as:

$$S_{FD}(y;\ k,\theta,\alpha,\varphi) = 1 - \left[1 - \frac{\{\Gamma(k,\theta y)+\,(\theta\alpha)^k e^{-\theta y}\}}{\{\,(\theta\alpha)^k+\Gamma k\}}\right]^{\varphi} \tag{6.1}$$

Hazard function:

Let Y be a continuous random variable with pdf. $f_{FD}(y;\ k,\theta,\alpha,\varphi)$ (2.5) and c.d.f. $F_{FD}(y;\ k,\theta,\alpha,\varphi)$ (2.4) of FD. The hazard rate function known as the failure rate function defined as:

$$h_{FD}(\mathrm{y}) = \lim_{\Delta y\to 0}\frac{P(Y<y+\Delta y|Y>y)}{\Delta y} = \frac{f(y)}{1-F(y)} \tag{6.2}$$

By using (2.4) and (2.5) find as:

$$h_{FD}(y;\ k,\theta,\alpha,\varphi) = \frac{\varphi\,\theta^k(\,\theta\alpha^k+y^{k-1})e^{-\theta y}[\{\,(\theta\alpha)^k+\Gamma k\}-\{\Gamma(k,\theta y)+\,(\theta\alpha)^k e^{-\theta y}\}]^{\varphi-1}}{\{\,(\theta\alpha)^k+\Gamma k\}^{\varphi}-[\{\,(\theta\alpha)^k+\Gamma k\}-\{\Gamma(k,\theta y)+\,(\theta\alpha)^k e^{-\theta y}\}]^{\varphi}} \tag{6.3}$$

Here note that:
$h_{FD}(0;\alpha,\theta) = f_{FD}(0;\alpha,\theta)$
Cumulative hazard function:
Let Y be a continuous random variable with pdf $f_{FD}(y;\ k,\theta,\alpha,\varphi)$and c.d.f. $F_{FD}(y;\ k,\theta,\alpha,\varphi)$ of (FD) then the Cumulative hazard function defined as:

$$CH_{FD}(y;\ k,\theta,\alpha,\varphi) = -\ln|F(y;\ k,\theta,\alpha,\varphi)| \qquad (6.6)$$

By putting (2.4) we have,

$$CH_{FD}(y;\ k,\theta,\alpha,\varphi) = -\ln\left|1-\left[\frac{\{(\theta\alpha)^k+\Gamma k\}-\{\Gamma(k,\theta y)+(\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k+\Gamma k\}}\right]^{\varphi}\right| \qquad (6.7)$$

Reversed hazard function:
Let Y be a continuous random variable with pdf $f_{FD}(y;\ k,\theta,\alpha,\varphi)$ and c.d.f. $F_{FD}(y;\ k,\theta,\alpha,\varphi)$ of (FD) then the reversed hazard function defined as:

$$H_{FD}(\mathrm{y}) = \frac{f(y;k,\theta,\alpha,\varphi)}{F(y;k,\theta,\alpha,\varphi)} \qquad (6.8)$$

By putting (2.4) and (2.5) into (6.8) have found

$$H_{FD}(y;\ k,\theta,\alpha,\varphi) = \frac{\varphi\,\theta^k(\theta\alpha^k+y^{k-1})e^{-\theta y}}{\{(\theta\alpha)^k+\Gamma k\}-\{\Gamma(k,\theta y)+(\theta\alpha)^k e^{-\theta y}\}} \qquad (6.9)$$

Oder Statistics (OS)

The density function $f\left(y_{(i)}\right)$ of "ith" order statistics (i=1, 2, …, n) from independent and identically distributed (i.i.d), random variable $y_1, y_2, \dots y_n$. The order statistics say as $y_{(1)}, y_{(2)}, \dots y_{(n)}$ the function of order statistics defined as:

$$f\left(y_{(i)}\right) = \frac{f(y)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}[F(y)]^{i+j-1} \qquad (7.1)$$

$$f\left(y_{(i)}\right) = \frac{\varphi\,\theta^k(\theta\alpha^k+y^{k-1})e^{-\theta y}}{B(i,n-i+1)\{(\theta\alpha)^k+\Gamma k\}}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}\left[\frac{\{(\theta\alpha)^k+\Gamma k\}-\{\Gamma(k,\theta y)+(\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k+\Gamma k\}}\right]^{\varphi(i+j)-1}$$

Here $B(i,n-i+1)$ is the beta function. Here, present an expansion for the density function of FD (2.1), $f_{FD}(y;\ k,\theta,\alpha,\varphi)\ and$ cdf (2.2) $F_{FD}(y;\ k,\theta,\alpha,\varphi)$ into (6.1) have pdf
The ith order statistics c.d.f is:

$$F\left(y_{(i)}\right) = \sum_{j=0}^{n}\sum_{i=0}^{n-j}(-1)^i\binom{n}{j}\binom{n-j}{i}\left\{\frac{\{(\theta\alpha)^k+\Gamma k\}-\{\Gamma(k,\theta y)+(\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k+\Gamma k\}}\right\}^{\varphi(j+i)} \qquad (7.3)$$

For maximum order statistics put i=n, for minimum order statistics put i=1 in equation.

Renyi Entropy Measure

A popular entropy measure is Renyi entropy by Reyni (1961), an entropy of a random variable Y is a measure of the variation of uncertainty. Let Y is a continuous random variable having probability density function (FD) $f_{FD}(y;\ k,\theta,\alpha,\varphi)$, then Renyi entropy is defined as

$$T_{RE}(y) = \frac{1}{1-\delta}\log\left\{\int_0^\infty f(y)^\delta\,dy\right\} \qquad (8.1)$$

Let $f_{FD}(y;\ k,\theta,\alpha,\varphi)$ pdf of (FD) the Renyi entropy such that:

$$T_{RE}(y;\ k,\theta,\alpha,\varphi) = \frac{1}{1-\delta}\log\left\{\int_0^\infty f_{FD}(y;\ k,\theta,\alpha,\varphi)^\delta\,dy\right\}$$

$$= \frac{1}{1-\delta}\log\left\{\int_0^\infty \varphi^\delta\left[1-\frac{\{\Gamma(k,\theta y)+(\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k+\Gamma k\}}\right]^{\delta(\varphi-1)}\left[\frac{\theta^k(\theta\alpha^k+y^{k-1})e^{-\theta y}}{(\theta\alpha)^k+\Gamma k}\right]^\delta dy\right\}$$

$$= \frac{1}{1-\delta}\log\left[\frac{\theta^{\delta k+\delta}\alpha^{k\delta}}{\{(\theta\alpha)^k+\Gamma k\}^\delta}\int_0^\infty\left[1-\frac{\{\Gamma(k,\theta y)+(\theta\alpha)^k e^{-\theta y}\}}{\{(\theta\alpha)^k+\Gamma k\}}\right]^{\delta(\varphi-1)}\left(1+\frac{y^{k-1}}{\theta\alpha^k}\right)^\delta dy\right]$$

$$\varphi^{\delta}\left[1-\frac{\{\Gamma(k)+(\theta\alpha)^{k}e^{-\theta y}-\gamma(k,\theta y)\}}{\{(\theta\alpha)^{k}+\Gamma(k)\}}\right]^{(\varphi-1)\delta}=$$

$$\varphi^{\delta}\sum_{i,j,l=0}^{\infty}(-1)^{l}\binom{(\varphi-1)\delta}{i}\binom{i}{j}\binom{j}{l}\frac{\{\Gamma(k)\}^{i-j+l}(\theta\alpha)^{k(j-l)}e^{-(j)\theta y}(\theta y)^{(k)(l)}\sum_{p=0}^{\infty}D_{p}(\theta y)^{p}}{\{(\theta\alpha)^{k}+\Gamma(k)\}^{i}}$$

$$a_{p}=\frac{1}{\Gamma(k+p+1)};\ D_{0}=a_{0}^{l};\ \ D_{m}=\frac{1}{ma_{0}}\sum_{p=1}^{m}(pl-m+p)a_{p}D_{m-p}$$

$$\left(1+\frac{y^{k-1}}{\theta\alpha^{k}}\right)^{\delta}=\sum_{q=0}^{\infty}\binom{\delta}{q}y^{q(k-1)}$$

$$=\frac{1}{1-\delta}\log\left\{\int_{0}^{\infty}\frac{\varphi^{\delta}\sum_{i,j,l=0}^{\infty}(-1)^{l}\binom{\varphi\delta-\delta}{i}\binom{i}{j}\binom{j}{l}}{\frac{\theta^{\delta k+\delta-q+jk+p}\alpha^{k(\delta-q+j-l)}\{\Gamma(k)\}^{i-j+l}e^{-j\theta y}\sum_{p=0}^{\infty}D_{p}(y)^{lk+p}\sum_{q=0}^{\infty}\binom{\delta}{q}y^{q(k-1)}e^{-\theta\delta y}}{\{(\theta\alpha)^{k}+\Gamma(k)\}^{i+\delta}}}dy\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\int_{0}^{\infty}\frac{\varphi^{\delta}\sum_{i,j,l,q=0}^{\infty}(-1)^{l}\binom{\varphi\delta-\delta}{i}\binom{i}{j}\binom{j}{l}\binom{\delta}{q}\left[\{\Gamma(k)\}^{i-j+l}\sum_{p=0}^{\infty}\begin{matrix}\theta^{\delta k+\delta-q+jk+p}\alpha^{k(\delta-q+j-l)}\\ D_{p}(y)^{lk+p+q(k-1)}e^{-\theta(j+\delta)y}\end{matrix}\right]}{\{(\theta\alpha)^{k}+\Gamma(k)\}^{i+\delta}}dy\right\}$$

Put $\theta(j+\delta)y=t$

$$=\frac{1}{1-\delta}\log\left\{\frac{\varphi^{\delta}\sum_{i,j,l,q=0}^{\infty}(-1)^{l}\binom{\varphi\delta-\delta}{i}\binom{i}{j}\binom{j}{l}\binom{\delta}{q}\left[\{\Gamma(k)\}^{i-j+l}\sum_{p=0}^{\infty}\begin{matrix}\theta^{\delta k+\delta-q+jk+p}\alpha^{k(\delta-q+j-l)}D_{p}\\ \int_{0}^{\infty}t^{k+p+q(k-1)+1-1}e^{-t}\,dy\end{matrix}\right]}{\{\theta(j+\delta)\}^{k+p+q(k-1)}\{(\theta\alpha)^{k}+\Gamma(k)\}^{i+\delta}}\right\}$$

$$=\frac{1}{1-\delta}\log\left\{\frac{\varphi^{\delta}\sum_{i,j,l,q=0}^{\infty}(-1)^{l}\binom{\varphi\delta-\delta}{i}\binom{i}{j}\binom{j}{l}\binom{\delta}{q}\left[(\Gamma\mathrm{k})^{i-j+l}\sum_{p=0}^{\infty}\begin{matrix}\theta^{k(\delta+j-q-1)+\delta}\alpha^{k(\delta-q+j-l)}\\ D_{p}\Gamma(k+p+q(k-1)+1)\end{matrix}\right]}{(j+\delta)^{k+p+q(k-1)}\{(\theta\alpha)^{k}+\Gamma\mathrm{k}\}^{i+\delta}}\right\}\quad(8.2)$$

Maximum Likelihood Method

Let be a random sample $y_1, y_2, \dots y_n$ from $f_{FD}(y_i;\alpha,\theta,\varphi,k)$ (FD), then the Maximum Likelihood (ML) function defined as:

$$L(y;\ \alpha,\theta,\varphi,k)=\prod_{i=0}^{n}\ \varphi\left[1-\frac{\{\Gamma(k,\theta y)+(\theta\alpha)^{k}e^{-\theta y}\}}{\{(\theta\alpha)^{k}+\Gamma k\}}\right]^{\varphi-1}\frac{\theta^{k}(\theta\alpha^{k}+y^{k-1})e^{-\theta y}}{(\theta\alpha)^{k}+\Gamma k}\quad(9.1)$$

$$lnL(y;\alpha,\theta,\varphi,k)=nln\varphi+(\varphi-1)\sum_{i=0}^{n}\ln\left[(\theta\alpha)^{k}\left(1-e^{-\theta y_{i}}\right)+\gamma(k,\theta y_{i})\right]+nkln\theta+\sum_{i=0}^{n}\ln(\theta\alpha^{k}+y_{i}^{k-1})-\theta\sum_{i=0}^{n}y_{i}-n\varphi\ln\{(\theta\alpha)^{k}+\Gamma\mathrm{k}\}\quad(9.2)$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta}=(\varphi-1)\sum_{i=0}^{n}\frac{\left[(\alpha)^{k}k\,(\theta)^{k-1}\left(1-e^{-\theta y_{i}}\right)+(\alpha)^{k}(\theta)^{k-1}e^{-\theta y_{i}}+\frac{\partial\gamma(k,\theta y_{i})}{\partial\theta}\right]}{\left[(\theta\alpha)^{k}\left(1-e^{-\theta y_{i}}\right)+\gamma(k,\theta y_{i})\right]}+\frac{nk}{\theta}+\sum_{i=0}^{n}\frac{\alpha^{k}}{(\theta\alpha^{k}+y_{i}^{k-1})}-\sum_{i=0}^{n}y_{i}-\frac{n\varphi\,(\alpha)^{k}k\,(\theta)^{k-1}}{\{(\theta\alpha)^{k}+\Gamma\mathrm{k}\}}\quad(9.3)$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha}=(\varphi-1)\sum_{i=0}^{n}\frac{(\alpha)^{k-1}k\,(\theta)^{k}\left(1-e^{-\theta y_{i}}\right)}{\left[(\theta\alpha)^{k}\left(1-e^{-\theta y_{i}}\right)+\gamma(k,\theta y_{i})\right]}+\sum_{i=0}^{n}\frac{k\alpha^{k-1}}{(\theta\alpha^{k}+y_{i}^{k-1})}-\frac{n\varphi\,\alpha^{k-1}k\,(\theta)^{k}}{\{(\theta\alpha)^{k}+\Gamma\mathrm{k}\}}\quad(9.4)$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi}=\frac{n}{\varphi}+\sum_{i=0}^{n}\ln\left[(\theta\alpha)^{k}\left(1-e^{-\theta y_{i}}\right)+\gamma(k,\theta y_{i})\right]-n\ln\{(\theta\alpha)^{k}+\Gamma\mathrm{k}\}\quad(9.5)$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial k} = (\varphi-1)\sum_{i=0}^{n}\frac{\left[(\theta\alpha)^k \ln(\theta\alpha)\left(1-e^{-\theta y_i}\right)+\frac{\partial\gamma(k,\theta y_i)}{\partial k}\right]}{\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right]} + \sum_{i=0}^{n}\frac{\theta\alpha^k\ln(\alpha)+{y_i}^{k-1}\ln(y_i)}{\left(\theta\alpha^k+{y_i}^{k-1}\right)} - \frac{n\varphi\left\{(\theta\alpha)^k ln(\theta\alpha)+\frac{\partial\Gamma k}{\partial k}\right\}}{\left\{(\theta\alpha)^k+\Gamma k\right\}} \quad (9.6)$$

By putting (9.3), (9.4), (9.5) and (9.6) as $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\theta}=0$; $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\alpha}=0$, $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial k}=0$, $\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\varphi}=0$

The (9.3), (9.4), (9.5) and (9.6) natural log likelihood equations do not seem to be solved directly because they are not in closed form, so the Fisher's scoring method can be applied to solve.

$$\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha^2} = (\varphi-1)\sum_{i=0}^{n}\frac{\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right](\alpha)^{k-2}k(k-1)(\theta)^k\left(1-e^{-\theta y_i}\right)-\left\{(\alpha)^{k-1}k(\theta)^k\left(1-e^{-\theta y_i}\right)\right\}^2}{\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right]^2} + \sum_{i=0}^{n}\frac{\left(\theta\alpha^k+{y_i}^{k-1}\right)k\alpha^{k-1}-\left(k\alpha^{k-1}\right)^2}{\left(\theta\alpha^k+{y_i}^{k-1}\right)^2} - \frac{n\varphi\left\{(\theta\alpha)^k+\Gamma k\right\}\alpha^{k-1}k(\theta)^k-\left\{\alpha^{k-1}k(\theta)^k\right\}^2}{\left\{(\theta\alpha)^k+\Gamma k\right\}^2} \quad (9.7)$$

$$\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta^2} = (\varphi-1)\sum_{i=0}^{n}\frac{\left[\begin{array}{c}\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right]\left[\alpha^k(k-1)\theta^{k-1}\left\{\theta\left(k-ke^{-\theta y_i}+e^{-\theta y_i}\right)+e^{-\theta y_i}\right\}+\frac{\partial^2\gamma(k,\theta y_i)}{\partial\theta^2}\right]\\ -\left[\alpha^k k\,\theta^{k-1}\left(1-e^{-\theta y_i}\right)+\alpha^k\theta^{\,k-1}e^{-\theta y_i}+\frac{\partial\gamma(k,\theta y_i)}{\partial\theta}\right]^2\end{array}\right]}{\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right]^2} - \frac{nk}{\theta^2} - \sum_{i=0}^{n}\frac{\alpha^{2k}}{\left(\theta\alpha^k+{y_i}^{k-1}\right)^2} - \frac{n\varphi\left\{(\theta\alpha)^k+\Gamma k\right\}\alpha^{\,k}k(k-1)\theta^{\,k-2}-\left\{\alpha^k k\,\theta^{k-1}\right\}^2}{\left\{(\theta\alpha)^k+\Gamma k\right\}} \quad (9.8)$$

$$\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi^2} = \frac{-n}{\varphi^2} \quad (9.9)$$

$$\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k^2} = (\varphi-1)\sum_{i=0}^{n}\frac{\left[\begin{array}{c}\left\{(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right\}\left\{(\theta\alpha)^k\{\ln(\theta\alpha)\}^2\left(1-e^{-\theta y_i}\right)+\frac{\partial^2\gamma(k,\theta y_i)}{\partial k^2}\right\}\\ -\left\{(\theta\alpha)^k\ln(\theta\alpha)\left(1-e^{-\theta y_i}\right)+\frac{\partial\gamma(k,\theta y_i)}{\partial k}\right\}^2\end{array}\right]}{\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right]^2} + \sum_{i=0}^{n}\frac{\left(\theta\alpha^k+{y_i}^{k-1}\right)\left[\theta\alpha^k\{\ln\alpha\}^2+{y_i}^{k-1}\{\ln(y_i)\}^2\right]-\left\{\theta\alpha^k\ln\alpha+{y_i}^{k-1}\ln(y_i)\right\}^2}{\left(\theta\alpha^k+{y_i}^{k-1}\right)^2} - \frac{n\varphi\left\{(\theta\alpha)^k+\Gamma k\right\}\left\{(\theta\alpha)^k\{ln(\theta\alpha)\}^2+\frac{\partial^2\Gamma k}{\partial k^2}\right\}-\left\{(\theta\alpha)^k ln(\theta\alpha)+\frac{\partial\Gamma k}{\partial k}\right\}^2}{\left\{(\theta\alpha)^k+\Gamma k\right\}} \quad (9.10)$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial\alpha} = (\varphi-1)\sum_{i=0}^{n}\frac{\left[\begin{array}{c}\left\{(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right\}\left\{k\left(1-e^{-\theta y_i}\right)+e^{-\theta y_i}\right\}k\,\alpha^{k-1}\theta^{k-1}\\ -\alpha^{k-1}k\,\theta^k\left(1-e^{-\theta y_i}\right)\left\{\alpha^k k\,\theta^{k-1}\left(1-e^{-\theta y_i}\right)+(\alpha)^k\theta^{k-1}e^{-\theta y_i}+\frac{\partial\gamma(k,\theta y_i)}{\partial\theta}\right\}\end{array}\right]}{\left[(\theta\alpha)^k\left(1-e^{-\theta y_i}\right)+\gamma(k,\theta y_i)\right]^2} + \sum_{i=0}^{n}\frac{k\alpha^{k-1}{y_i}^{k-1}}{\left(\theta\alpha^k+{y_i}^{k-1}\right)^2} - \frac{n\varphi k^2(\theta\alpha)^{k-1}\left\{(\theta\alpha)^k+\Gamma k-\alpha^{k-1}k\,\theta^k\right\}}{\left\{(\theta\alpha)^k+\Gamma k\right\}} \quad (9.11)$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial k}=(\varphi-1)\sum_{i=0}^{n}\left[\frac{\left[\begin{array}{c}\{(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\}\times\\ \left[\{(\theta\alpha)^k\ln(\theta\alpha)k+(\theta\alpha)^k\}(\theta)^{-1}(1-e^{-\theta y_i})+(\theta\alpha)^k\ln(\theta\alpha)\,\theta^{-1}e^{-\theta y_i}+\frac{\partial\gamma(k,\theta y_i)}{\partial\theta\partial k}\right]\\ -\left\{(\theta\alpha)^k\ln(\theta\alpha)(1-e^{-\theta y_i})+\frac{\partial\gamma(k,\theta y_i)}{\partial k}\right\}\left\{\alpha^k k\,\theta^{k-1}(1-e^{-\theta y_i})+\alpha^k\,\theta^{k-1}e^{-\theta y_i}+\frac{\partial\gamma(k,\theta y_i)}{\partial\theta}\right\}\end{array}\right]}{\left[(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\right]^2}\right]+\frac{nk}{\theta}+$$

$$\sum_{i=0}^{n}\frac{\alpha^k}{(\theta\alpha^k+y_i{}^{k-1})}-\sum_{i=0}^{n}y_i-\frac{n\varphi\,\alpha^k k\,\theta^{k-1}}{\{(\theta\alpha)^k+\Gamma k\}} \tag{9.12}$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha\partial k}=(\varphi-1)\sum_{i=0}^{n}\frac{\left[\begin{array}{c}\{(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\}\{(\alpha)^{-1}k\,(\theta\alpha)^k\ln(\theta\alpha)+\alpha^{-1}(\theta\alpha)^k\}(1-e^{-\theta y_i})-\\ \left\{(\theta\alpha)^k\ln(\theta\alpha)(1-e^{-\theta y_i})+\frac{\partial\gamma(k,\theta y_i)}{\partial k}\right\}\alpha^{k-1}k\,\theta^k(1-e^{-\theta y_i})\end{array}\right]}{\left[(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\right]^2}+$$

$$\sum_{i=0}^{n}\frac{(\theta\alpha^k+y_i{}^{k-1})\{k\alpha^{k-1}ln\alpha+\alpha^{k-1}\}-(\theta\alpha^k ln\alpha+y_i{}^{k-1}lny_i)k\alpha^{k-1}}{(\theta\alpha^k+y_i{}^{k-1})^2}-$$

$$\frac{n\varphi\{(\theta\alpha)^k+\Gamma k\}\{\alpha^{-1}k\,(\theta\alpha)^k\ln(\theta\alpha)+\alpha^{-1}(\theta\alpha)^k\}-n\varphi\left\{(\theta\alpha)^k\ln(\theta\alpha)+\frac{\partial\Gamma k}{\partial k}\right\}\alpha^{k-1}k\,\theta^k}{\{(\theta\alpha)^k+\Gamma k\}^2} \tag{9.13}$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial\theta}=\sum_{i=0}^{n}\frac{\left[\alpha^k k\,\theta^{k-1}(1-e^{-\theta y_i})+\alpha^k\,\theta^{k-1}e^{-\theta y_i}+\frac{\partial\gamma(k,\theta y_i)}{\partial\theta}\right]}{\left[(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\right]}-\frac{n\,\alpha^k k\,\theta^{k-1}}{\{(\theta\alpha)^k+\Gamma k\}} \tag{9.14}$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial\alpha}=\sum_{i=0}^{n}\frac{\left[(\theta\alpha)^k\ln(\theta\alpha)(1-e^{-\theta y_i})+\frac{\partial\gamma(k,\theta y_i)}{\partial k}\right]}{\left[(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\right]}-\frac{n\,\alpha^{k-1}k\,(\theta)^k}{\{(\theta\alpha)^k+\Gamma k\}} \tag{9.15}$$

$$\frac{\partial lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial k}=\sum_{i=0}^{n}\frac{\left[(\theta\alpha)^k\ln(\theta\alpha)(1-e^{-\theta y_i})+\frac{\partial\gamma(k,\theta y_i)}{\partial k}\right]}{\left[(\theta\alpha)^k(1-e^{-\theta y_i})+\gamma(k,\theta y_i)\right]}-\frac{n\left\{(\theta\alpha)^k ln(\theta\alpha)+\frac{\partial\Gamma k}{\partial k}\right\}}{\{(\theta\alpha)^k+\Gamma k\}} \tag{9.16}$$

$\left\{\frac{d}{dk}\Gamma k\right\}$ *is digmma function*

The iterative solution of the equations (9.7) to (9.16) using matrix given following will be the MLEs $\hat{\theta}$ $\hat{\alpha}$, $\hat{\varphi}$ *and* $\hat{k}$of parameters θ, α, φ and k of FD.

$$\left[\begin{array}{cccc}\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta^2} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k\partial\theta} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial\theta}\\ \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha^2} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k\partial\alpha} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial\alpha}\\ \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial k} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha\partial k} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k^2} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial k}\\ \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial\varphi} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha\partial\varphi} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k\partial\varphi} & \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi^2}\end{array}\right]_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{k}=k_0\\ \hat{\varphi}=\varphi_0}}$$

$$\begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{k}-k_0 \\ \hat{\varphi}-\varphi_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL(y;\,\theta,\alpha,k)}{\partial\theta} \\ \frac{\partial lnL(y;\,\theta,\alpha,k)}{\partial\alpha} \\ \frac{\partial lnL(y;\,\theta,\alpha,k)}{\partial k} \\ \frac{\partial lnL(y;\,\theta,\alpha,k)}{\partial\varphi} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{k}=k_0 \\ \hat{\varphi}=\varphi_0}}$$

We know that; $\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial k} = \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k\partial\varphi}$; $\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial\varphi} = \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial\theta}$; $\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha\partial\varphi} = \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\varphi\partial\alpha}$; $\frac{\partial^2 lnL^2(y;\,\theta,\alpha,k)}{\partial\alpha\partial k} = \frac{\partial^2 lnL^2(y;\,\theta,\alpha,k)}{\partial k\partial\alpha}$; $\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial\alpha} = \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\alpha\partial\theta}$; $\frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial\theta\partial k} = \frac{\partial^2 lnL(y;\alpha,\theta,\varphi,k)}{\partial k\partial\theta}$; also θ_0, α_0, k_0 and φ_0 are initial values of parameters of θ, α, k, and φ of FD respectively.

Application

Data set 1. The data set is the strength data of glass of the aircraft window reported by (Fuller *et al.*,1994) and are given as 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Data set 2. By Lawless (1982), the data given arose in tests on endurance of deep groove ball bearings and data are the number of million revolutions before failure for each of the 23 ball bearings in the life tests and they are following: 17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

In order to compare lifetime distributions, K-S Statistics (Kolmogorov-Smirnov Statistics), Person Chi square and Anderson Darling statistics for the above data set have been computed their statistics defined respectively.

Kolmogorov-Smirnov statistics:

$K-S = Sup_{(y)}|F_n(y) - F_0(y)|$

$F_n(y)$= empirical distribution, n= sample size

Anderson Darling statistics:

$A^2 = -n - S$

Where $S = \sum_{i=1}^{n}(2i-1)/n[lnF(y_i) + \ln\{1 - F(y_{n+1-i})\}]$

F is the cumulative distribution function of the specified distribution and y_i are ordered data.

Cramer Von Mises statistics:

Let $y_1, y_2, \dots y_n$ be the (independent and identically-distributed) random variable have continuous distribution with cdf $F(y)$, the Cramér–von Mises test statistic based on following statistic

$$\omega_n^2[\psi(F(y))] = \int_{-\infty}^{+\infty}\left[\sqrt{n}\{F_n(y) - F(y)\}\right]^2 \psi(F(y))dF(y)$$

Here $F_n(y)$is the empirical distribution function constructed from the sample size n and $\psi(F(y))$ is a certain nonnegative function defined and integrated on the interval [0, 1]. Simplified as;

$$\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(y_{(i)}) - \frac{2i-1}{2n}\right]^2$$

Here $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ is the varied series based on the sample $y_1, y_2, \dots y_n$.

Table 1. The MLE's, K-S statistics, Anderson-Darling and Cramer Von Mises statistics values using data set 1

Distributions	MLE's				K-S statistics	Anderson -Darling	Cramér-von Mises
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$	$\hat{\varphi}$			
FD	0.31329	0.12623	6.52459	4.849	0.12521	0.41508	0.0796
Ram Awadh	0.19473				0.19768	2.00445	0.3026
Suja	1.65272				0.22321	2.54007	0.4048
Pranav	0.12978				0.25346	3.24745	0.5479
Akash	0.09706				0.29463	4.19329	0.7488
Three PLD	0.06489	0.0210	2.00		0.35866	5.69565	1.0985
TPLD	0.06488	0.0100			0.35865	5.69540	1.0984
Lindley	0.06298				0.36545	5.87165	1.1389
Exponential	0.03245				0.45862	8.52724	1.7882

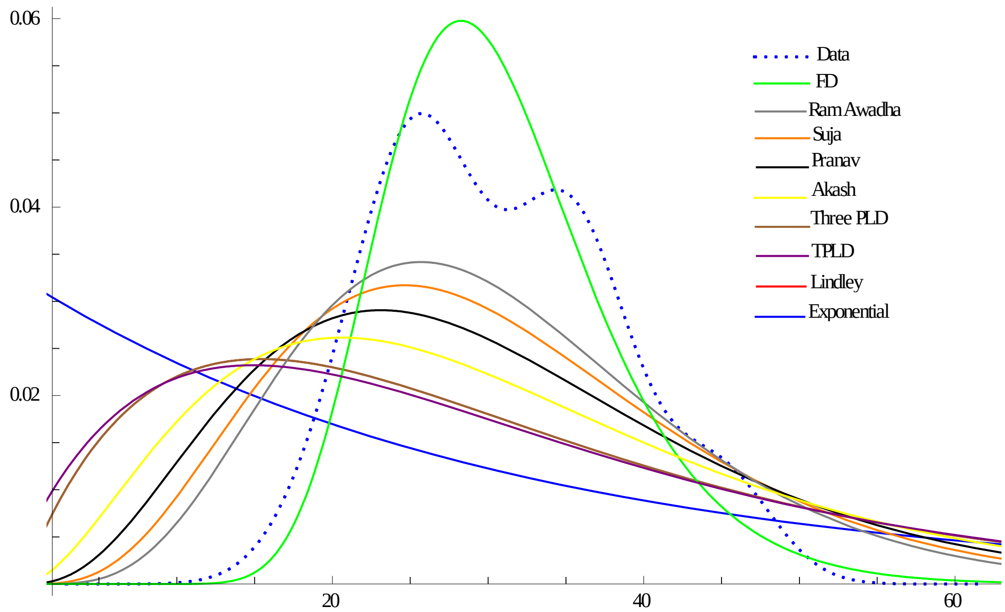

Figure 1. Goodness of fit of distributions with data set 1

Table 2. The MLE's, K-S statistics, Anderson-Darling and Cramer Von Mises statistics values using data set 2

Distributions	MLE's				K-S statistics	Anderson -Darling	Cramér-von Mises
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$	$\hat{\varphi}$			
FD	0.03229	10007.8	226.6	5.28	0.10579	0.18710	0.03219
Ram Awadh	0.08306				0.16138	0.69845	0.10593
Suja	0.06922				0.13715	0.38168	0.06505
Pranav	0.05537				0.12269	0.21403	0.03884
Akash	0.04151				0.11834	0.29785	0.04633
Three PLD	0.02768	0.0000015	559.1		0.18862	0.87705	0.14034
TPLD	0.02768	0.0000002			0.18862	0.87705	0.14034
Lindley	0.01476				0.42306	7.44291	1.58167
Exponential	0.01384				0.30676	2.81422	0.53689

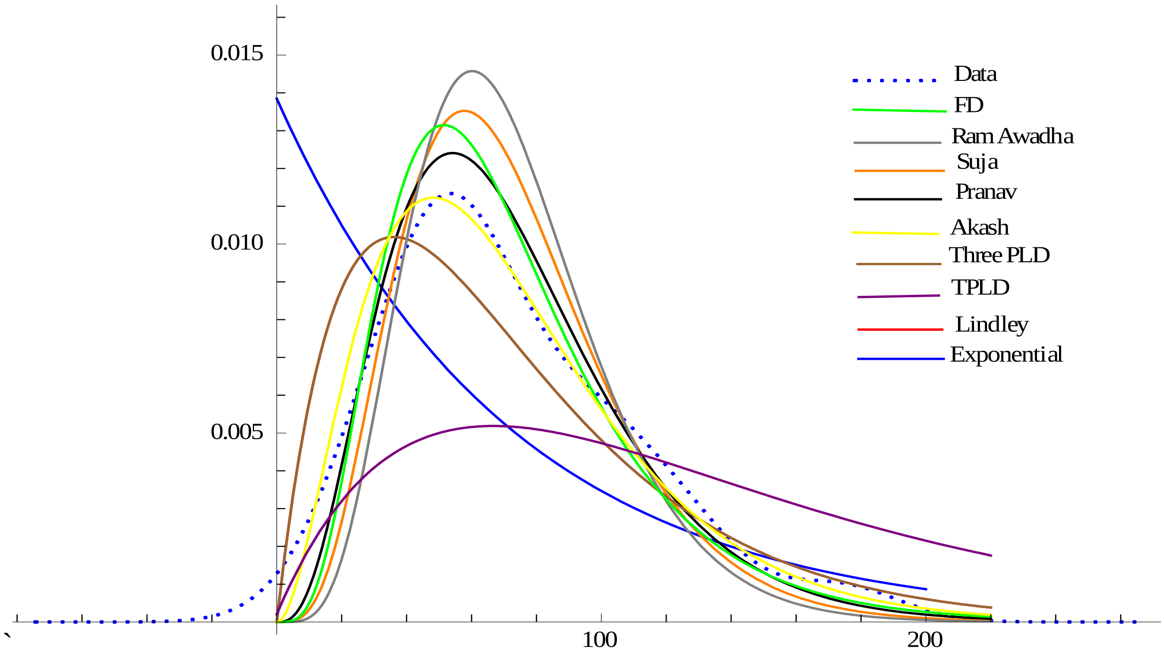

Figure 2. Goodness of fit of distributions with data set 2

Conclusion

The new proposed Faisal distribution is a life time data analysis distribution is a complete pdf with four parameters are α, θ, φ, and k. if $\varphi = 1$; then for the value of k=1 it reduced to exponential distribution, k=2; $\alpha = 1$ is it is one parameter Shanker distribution, $k = 3, \alpha = 1$ is Ishtia distribution, $k = 4$; $\alpha = 1$ it is Pranav distribution, $k = 5$; $\alpha = 1$ is Rani Distribution, $k = 6$; $\alpha = 1$ is Ram Awadha distribution, $\alpha = 0$ it reduced to $gamma\ (k, \theta)$. Further discussed properties including moment generating function of FD, rht moment, first four moment about origin, mean deviations about mean and median, Bonnferoni and Lorenz curve, reliability measures (survival function, hazard function, cumulative hazard function, reversed hazard function), order statistics, Reyni entropy also likelihood estimation method. From Table 1 observed the FD distribution have K-S statistics, Anderson-Darling and Cramer Von Mises statistics values minimum then other available lifetime distributions also from Table 2 similar results obtained. The graphical representation of goodness of fit with data showing in Figure 1 and Figure 2, better fitness of FD then other distributions with real life data applications.

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References

Ahmad, Z., Hamedani, G. G., & Butt, N. S. (2019). Recent developments in distribution theory: a brief survey and some new generalized classes of distributions. *Pakistan Journal of Statistics and Operation Research*, *15*, 87–110.

Bonferroni, C. E. (1930). Elementi di Statistca generale. *Seeber, Firenze*.

Fuller, E. J., Frieman, S., Quinn, J., Quinn, G., & Carter, W. (1994). Fracture mechanics approach to the design of glass. *International Society for Optical Engineering*, *2286*, pp. 419-430.

Gradshetyn, I. S., & Ryzhik, I. M. (2000). *Tables of Integrals, Series and Products* (6th ed.). San Diego: Academic Press.

Lawless, J.F. (1982). *Statistical models and methods for lifetime data*. New York: John Wiley and Sons.

Lindley, D. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B, 20*, 102-107.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE transactions on reliability*, *42*(2), 299-302.

Mussie, T., & Shanker, R. (2018). A two-Parameter Sujatha distribution. *Biometrics Biostat. Internat. J.*, *7*(3), 188–197.

Renyi, A. (1961). On measures of entropy and information. *Proceedings of the 4th Berkeley symposium on Mathematical Statistics and Probability* (pp. 547-561.). Berkeley.

Shanker, R. (2015a). Shanker Distribution and Its Applications. *International Journal of Statistics and Applications*, *5*(6), 338–348.

Shanker, R. (2016). A quasi Sujatha distribution. *Internat. J. Probab. Stat.*, *5*(4), 89–100.

Shanker, R. (2017a). Rama distribution and its Application. *International Journal of Statistics and Applications*, *7*(1), 26-35.

Shanker, R. (2017). Rani Distribution and Its Application. *Biometrics & Biostatistics International Journal*, *6*(1), 1–10.

Shanker, R. (2017). Suja Distribution and Its Application. *International Journal of Probability and Statistics*, *6*(2), 11-19.

Shanker, R. & Shukla, K. K. (2017). Ishita distribution and its Applications. *Biometrics & Biostatistics International Journal*, *5*(2), 1–9.

Shanker, R., & Shukla, K. K. (2019). Rama-Kamlesh distribution and Its Applications. *International Journal of Engineering and Future Technology*, *16*(4), 11-22.

Shanker, R., Sharma S., & Shanker, R. (2013). A two-parameter Lindley distribution for modeling waiting and survival times data. *Applied Mathematics*, *4*, 363-368.

Shukla, K. K. (2018). Pranav distribution with properties and its applications. *Biometrics & Biostatistics International Journal*, *7*(3), 244–254.

Shukla, K. K. (2018). Ram Awadh distribution with properties and applications. *Biometrics & Biostatistics International Journal*, *7*(6), 515–523.