A Necessary and Sufficient Condition for Matrix Solution $\Phi(t)$ of $\dot{\Phi}(t) = A(t)\Phi(t)$ to be a Fundamental Matrix

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Abstract. In this paper, the role played by fundamental matrix in the determination of solution $x(t)$ of the control systems of the form $\dot{x}(t) = A(t)x(t) + b(t)$ was ex-rayed. A necessary and sufficient condition for the matrix solution $\Phi(t)$ of $\dot{\Phi}(t) = A(t)\Phi(t)$ to be a fundamental matrix was also given.

Key words: Homogeneous system, Linear independent, Fundamental matrix, Trace of a matrix

Introduction

Let us consider the linear system of the form

$$\dot{x}(t) = A(t)x(t), t \in I = [0, \infty) \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state and $A(t)$ is an $n \times n$ continuous matrix function. Equation (1.1) is called a linear homogeneous system of order n. We note that the zero vector function on $I = [0, \infty)$ is always a solution to equation (1.1) (Lee & Markus, 1967). We are highly interested on the non - trivial solution of the equation. In part 2 of this work, we gave the literature review of the authors who made remarkable use of the fundamental matrix. Also, in part 3, we gave some propositions and Lemmas which helped us to achieve the major aim of this paper. In the last part of this paper, we gave the major result, followed by our recommendations. Let us start by introducing the symbols we used throughout this paper.

Notations:

$\Phi(t)$ = the fundamental matrix.
$\varphi(t)$ = the assumed solution of equation (1.1)
$tr.A$ = the trace of the square matrix A.

Let us state the definitions of some of the important terms we used in this paper.

Definitions:

Definition 1.1

The set of linearly independent solutions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$, on the interval $I = [0,\infty)$ are called the fundamental set of the solution of (1.1).

Definition 1.2

The matrix $\Phi(t)$, whose columns are the n - linearly independent solutions of (1.1) on the interval $I = [0,\infty)$ is called the fundamental matrix for (1.1).

Definition 1.3 Trace of $n \times n$ matrix A

The Trace of $n \times n$ matrix A, denoted by $tr.A$, is the sum of the diagonal terms of the matrix A. i.e. $tr.A = a_{11} + a_{22} + \ldots a_{nn}$.

Review of Literature

Fundamental matrix $\Phi(t)$ plays a great role in the determination of solutions to linear systems. Barnett (1975) defined the fundamental matrix $\Phi(t,t_0)$ as the component in the solution of linear system $\dot{x} = Ax + Bu$, subject to the initial conditions $x(t_0) = x_0$. That is

$$x(t) = \Phi(t, t_0)x_0 + \Phi(t, t_0)\int_{t_0}^{t} \Phi^{-1}(t, s)B(s)u(s)ds,$$

where $\Phi(t, t_0) = e^{[A(t-t_0)]}$ and $\Phi(0,0) = I$ the identity matrix.

Chi-Tsong (1984) in his own definition explained that an $n \times n$ matrix function $\Phi(t)$ is said to be fundamental matrix of $\dot{x} = A(t)x$ if and only if the n columns of Φ consists of n linearly independent solution of $\dot{x} = A(t)x$. Eke (1990) considered the null controllability of system of the form

$$\begin{cases} \dot{x} = A(t)x(t) + B(t)u(t) \\ x(t_0) = x_0 \end{cases} \quad (2.1)$$

He concluded that, by the variation of parameters, the solution of (2.1) above was given by

$$x(t,u) = \mathrm{X}(t)x_0 + \mathrm{X}(t)\int_O^t \mathrm{X}^{-1}(s)B(s)u(s)ds. \quad (2.2)$$

where $X(t)$ is the fundamental matrix of the system (2.1), for $B = 0$ with $X(0) = I$, the identity matrix.

The solution of (2.1) according to Katsuhisa et al. (1988) was given by

$$x(t) = e^{At}x_O + \int_O^t \mathrm{e}^{A(t-s)} B(s)u(s)ds. \quad (2.3)$$

where the function e^{At} is the fundamental matrix.

In the remark by Aniaku (2011) in his own paper, he said that the unique solution of the linear equation (2.1) was given by

$$x(t) = \Phi(t_, t_0)[x_0 + \int_{t_0}^t \Phi^{-1}(t_, s)B(s)u(s)ds] \quad (2.4)$$

where $\Phi(t,s)$ is the fundamental matrix satisfying

$$\dot{\Phi}(t_, t_0) = A(t)\Phi(t_, t_0) \text{ for } t \geq 0$$

and $\Phi(0,0) = I$; the identity matrix.

Silverman and Meadows (1967) explained that the output of the system

$$\begin{cases} \dot{x} = A(t)x(t) + B(t)u(t) \\ \mathcal{Y}(t) = C(t)x(t) \end{cases} \quad (2.5)$$

is given by

$$y(t) = C(t)X(t)[X^{-1}(t_0)x + \int_{t_0}^t \mathrm{X}^{-1}(\tau)B(\tau)u(\tau)d\tau] \quad (2.6)$$

where $x(t_0)$ is the state of the system at some arbitrary time t_0 and $X(t)$ is a fundamental matrix for the homogeneous part of (2.5) and $X(t)$ is non-singular for all t.

Interested studies can be obtained in the works of Watkins (2002), Chen (2007), Johansson (2017), Li and Yuan (2018), Kanatani (2009), Feng (2012), Ben-Artzi et al. (2016), Barragan, Trujillo, and Cabezas (2015), Hui (2007).

Lemmas and Propositions

We hereby state some of the lemmas and propositions which helped us to achieve the goal of this paper.

Lemma 3.1
The fundamental matrix $\Phi(t)$ satisfies the matrix equation

$$\dot{\Phi}\,(t) = A\Phi(t), t \in I = [0, \infty) \quad (3.1)$$

We also state lemma 3.2 thus:
Lemma 3.2
The set of all solutions of (1.1) on $I = [0,\infty)$ form an n - dimensional vector space over the complex field.
Proof:
The solution form a vector space. For if ϕ_1, ϕ_2 are solutions of (1.1) and c_1,c_2 are complex numbers, then $c_1\phi_1 + c_2\phi_2$ is again a solution of (1.1).
To show that the space is $n-$ dimensional, a set of n linearly independent solutions $\varphi_1, \varphi_2,...\phi_n$ must be exhibited such that every other solution is a linear combination (with complex coefficients) of these φ_i.
Now let ξ_i ($i = 1,2,...,n$) be n linearly independent points in the n-dimensional space. For instance ξ_i may be taken to be the vector with all components zero except the i^{th}, which is 1.
By the existence theorem, if $\tau \in I$, there exist n solutions φ_i,($i = 1,2,...,n$) of (1.1) such that $\varphi_i(\tau) = \psi_1$. We now show that these solutions ($\varphi_1, \varphi_2,..., \varphi_n$) satisfy the required conditions.
Firstly, they are linearly independent. For suppose the φ_{i}' are linearly dependent. Then the complex numbers $c_1,c_2,...,c_n$ not all zero exist such that

$$\sum_{i=1}^{n} c_i\varphi_i(t) = 0 \;\forall t \in I$$

This implies, in particular that

$$\sum_{i=1}^{n} c_i\varphi_i(\tau) = \sum_{i=1}^{n} c_i\xi_i(t) = 0$$

This contradicted the assumption that the $\xi_{\mathrm{i}}'s$ are linearly independent.
The following proposition assessed the status of the product of constant $n \times n$ matrix C with fundamental matrix Φ.
Proposition 3.1
If Φ is a fundamental matrix of (1.1) and C a constant non singular matrix, then ΦC is again fundamental matrix of (1.1).
Proof:
By Lemma 3.2, if Φ is a fundamental matrix, then

$$\Phi'\,(t)C = A(t)\Phi\,(t)C, t \in I \quad (3.2)$$

or

$$(\Phi C)' = A(\Phi C),$$

and hence, ΦC is a solution matrix of

$$X' = A(t)X, t \in I \quad (3.3)$$

So, the matrix Φ is called a solution of (3.2) on I and Φ satisfies (3.1). Since $det(\Phi C) = (det\Phi)(detC) \neq 0$, so ΦC is a fundamental matrix.
The following lemma assessed the relationship between a fundamental matrix Φ_2 and the product of $n \times n$ matrix C with fundamental matrix Φ_1.
Lemma 3.3
If Φ_1. and Φ_2 are fundamental matrices of (1.1), then $\Phi_2 \equiv \Phi_1 C$, for some non-trivial $n \times n$ matrix C.

Proof:
Let $\Phi_1^{-1}\Phi_2 = \Psi$ Then, $\Phi_2 = \Phi_1\Psi$.
Differentiating this, we get

$$\dot{\Phi}_2 = \Phi_1\Psi' + \Phi_1'\Psi$$

Then using the fact that $\Phi' = A\Phi$, we get

$$A\Phi_2 = \Phi_1\Psi' + A\Phi_1\Psi$$

$\Rightarrow \Phi_1\Psi' = 0 \Rightarrow \Phi' = 0$
So, we have that $\Psi = C$, a constant.
This constant C is non-singular since Φ_1 and Φ_2 are non-singular.
[Also, $det\Psi = det(\Phi - 1).det(\Phi 2) \neq 0$]
The following proposition warned us against assuming that the product of any $n \times n$ matrix C with a fundamental matrix Φ is a fundamental matrix.

Proposition 3.2
If Φ is a fundamental matrix of (1.1) and C any complex non-singular matrix, then ΦC is a fundamental matrix, but $C\Phi$ is not generally a fundamental matrix.

Proof:
We have seen that ΦC is a fundamental matrix. But for $C\Phi$ to be a fundamental matrix, it must satisfy two conditions simultaneously viz: (1) $det(C\Phi) \neq 0$ and (2) $C\Phi$ must satisfy $\dot{\Phi}(t) = A\Phi(t)$.
For (1) $det(C\Phi) \neq 0$ since neither C nor Φ is singular.
(2) If $C\Phi$ satisfies $\Phi'(t) = A(t)\Phi(t)$, then $C\Phi' = AC\Phi$ or $\Phi' = C^{-1}AC\Phi$, which is not the given differential equation unless A and C commute. So, $C\Phi$ can only be a fundamental matrix if A and C commute. This implies that $C\Phi$ is not generally a fundamental matrix.
The following proposition informed us on how we can get a solution function of equation (1.1).

Proposition 3.3 (Lee & Markus, 1967)
If Φ is a fundamental matrix of (1.1), then the function $\Phi(t)$ defined by

$$\varphi(t) = \Phi(t)\int_{\tau}^{t}\Phi^{-1}(s)b(s)ds \tag{3.4}$$

is the solution of the equation

$$\begin{cases} \dot{x} = A(t)x(t) + b(t) \\ x(\tau) = 0 \end{cases} \tag{3.5}$$

Proof:
Let $\phi = \Phi C$ be a solution of (3.5) where C is assumed for the moment to be some function of t. Differentiating, we get

$$\acute{\varphi}(t) = \acute{\Phi}(t)C + \Phi(t)C' \tag{3.6}$$

Φ is fundamental matrix $\Longrightarrow \Phi(t)$ satisfies

$$\varphi'(t) = A(t)\Phi(t) \tag{3.7}$$

Then (3.6) becomes on substitution of (3.7)
$\varphi'(t) = A(t)\Phi(t)C + \Phi(t)C'$
$= A\varphi(t) + \Phi(t)C'$ (Since $\varphi = \Phi C$, say)
$= A\varphi(t) + b(t)$,[provided $b(t) = \Phi C'$]

$$i.e\ C' = \Phi^{-1}b. \tag{3.8}$$

Integrating (3.8) from τ to t on both sides, we get

$$C(t) - C(\tau) = \int_\tau^t \Phi^{-1}(s)b(s)ds. \quad (3.9)$$

If $C(\tau) = 0$ in (3.9), we get

$$C(t) = \int_\tau^t \Phi^{-1}(s)b(s)ds.$$

So, since $\varphi = \Phi C$, we get

$$\varphi(t) = \Phi(t)\int_\tau^t \Phi^{-1}(s)b(s)ds.$$

as required.

Proposition 3.4

Let $[\tau, t] \subseteq I$, then

$$det\Phi(t) = det\Phi(\tau)e^{(\int_\tau^t tr.A(s))ds} \quad (3.10)$$

Proof:

Let us consider the matrix differential equation

$$\Phi'(s) = A(s)\Phi(s) \quad (3.11)$$

Let $\phi_{ij}(s)$ be the element in the i^{th} row and $\alpha_{ij}(s)$ be the elements in the j^{th} column of A. We take that $\Phi(s)$ and $A(s)$ each is an $n \times n$ matrix. From (3.11) and taking into account rule for matrix multiplication, we get

$$\varphi'_{ij}(s) = \sum_{k=1}^{n} \alpha_{ik}(s)\varphi_{kj}(s) \quad (3.12)$$

(i,j = 1,2, …,n)

Now,

$$det(\Phi) = |\Phi| = \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdot & \cdot & \cdot & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdot & \cdot & \cdot & \varphi_{2n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \varphi_{n1} & \varphi_{n2} & \cdot & \cdot & \cdot & \varphi_{nn} \end{vmatrix} \quad (3.13)$$

Differentiating (3.13), we get

$$(\det\Phi)' = \begin{vmatrix} \varphi'_{11} & \varphi'_{12} & \cdot & \cdot & \cdot & \varphi'_{1n} \\ \varphi_{21} & \varphi_{22} & \cdot & \cdot & \cdot & \varphi_{2n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \varphi_{n1} & \varphi_{n2} & \cdot & \cdot & \cdot & \varphi_{nn} \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdot & \cdot & \cdot & \varphi_{1n} \\ \varphi'_{21} & \varphi'_{22} & \cdot & \cdot & \cdot & \varphi'_{2n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \varphi_{n1} & \varphi_{n2} & \cdot & \cdot & \cdot & \varphi_{nn} \end{vmatrix} + \dots + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdot & \cdot & \cdot & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdot & \cdot & \cdot & \varphi_{2n} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \varphi'_{n1} & \varphi'_{n2} & \cdot & \cdot & \cdot & \varphi'_{nn} \end{vmatrix} \quad (3.14)$$

Substituting (3.12) into (3.14), we get

$$(\det\Phi)' = \begin{vmatrix} \sum_k \propto_{1k} \varphi k1 & \sum_k \propto_{1k} \varphi k2 & \cdots & \sum_k \propto_{1k} \varphi kn \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & & & \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \sum_k \propto_{2k} \varphi k1 & \sum_k \propto_{2k} \varphi k2 & \cdots & \sum_k \propto_{2k} \varphi kn \\ \vdots & & & \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & & & \\ \sum_k \propto_{nk} \varphi k1 & \sum_k \propto_{nk} \varphi k2 & \cdots & \sum_k \propto_{nk} \varphi kn \end{vmatrix} \quad (3.15)$$

From first component of (3.15), subtract first row P where
$P = (\alpha_{12} \times 2nd\ row + \alpha_{13} \times 3rd\ row + \ldots + \alpha_{1n} \times n^{th}\ row)$
From 2nd component of (3.15), subtract second row Q where
$Q = (\alpha_{11} \times 1st\ row + \alpha_{13} \times 3rd\ row + \ldots + \alpha_{1n} \times n^{th}\ row)$
From 3rd component of (3.15), subtract ... etc. These operations leave the values of the right hand side of (3.15) unchanged.
So, its form is
$\alpha_{11}\, det\Phi + \alpha_{22}\, det\Phi + \alpha_{33}\, det\Phi + \ldots + \alpha_{nn}\, det\ \Phi$
ie $(det\ \Phi) = (\alpha_{11} + \alpha_{22} + \ldots + \alpha_{nn}) det\Phi$
$= tr.\ A(s)\ det\Phi.$

$$\Rightarrow \frac{(\det\Phi)'}{\det\Phi} = tr.A(s). \quad (3.16)$$

Integrating both sides of (3.16) from τ to t, we get

$$\int_\tau^t \frac{det\ \Phi\acute{)}}{\det\Phi\,(s)} ds = \int_\tau^t tr.A(s)ds$$

$$\Longrightarrow loge\,(\det\Phi\,(s))\mathrm{I}_\tau^t = \int_\tau^t tr.A(s)ds$$

$$\Longrightarrow loge\,\frac{det\ \Phi(t))}{\det\Phi\,(\mathcal{T})} = \int_\tau^t tr.A(s)ds$$

$$\Longrightarrow \det\Phi = \det\Phi(\mathcal{T})e^{\int_\tau^t tr..A(s)}ds$$

as required.
Note that the following proposition assured us that the fundamental matrix of a homogeneous linear systems is the same as the exponential matrix of the matrix A, provided that A is a constant $n \times n$ matrix.
Proposition 3.5
A fundamental matrix for

$$\dot{x}(t) = Ax(t), t \in [0, \infty) \quad (3.17)$$

With A a constant $n \times n$ matrix is given by $\Phi(t) = e^{tA}$

Proof:

First of all, $\Phi = e^{tA}$ satisfies $\Phi' = A\Phi$, Since $Ae^{tA} = Ae^{tA}$. Then, from proposition 3.4, we have that

$$\det(e^{At}) = \det(I)e^{(\int_0^t tr.A(s)}ds \quad (3.18)$$

and the trace of A are all constants as A is a constant matrix.

$det(e^{tA}) = det(I)e^{\{(a_{11}+a_{22}+\ldots+a_{nn})t\}}$

$= e^{t(a_{11}+a_{22}+\ldots+a_{nn})}$

$\neq 0.$ (3.18)

So e^{At} is a fundamental matrix.

Main Result

We hereby state the main result of this paper.

Theorem 4

The necessary and sufficient condition that a solution matrix Φ of $\Phi'(t) = A(t)\Phi(t)$ is a fundamental matrix is that $det\,\Phi(t) \neq 0$ for $t \in I$, a real interval.

Proof.

Let Φ be a fundamental matrix with column vectors $\varphi_j, j = 1,2,.,n$. Suppose φ is any non-trivial solution of (1.1).By *lemma* 3.2, there exist unique constants $c_1,c_2,\ldots,c_n$ not all zero such that

$$\varphi = \sum_{j=1}^{n} c_j \varphi i$$

or in terms of the fundamental matrix Φ

$$\varphi = \Phi C \quad (4.1)$$

Where $c = c_1, c_2, c_3, \ldots, c_n$, n –liner equations in the unknowns $c_1, c_2, c_3, \ldots, c_n$ at $t_0 \in I$. This has a unique solution for any choice of $\varphi(t_0)$. Hence $det\Phi(t_0) \neq 0$ and $det\Phi(t_0) \neq 0\ \forall t \in I \implies col\ (c = c_1, c_2, c_3, \ldots, c_n)$ of the fundamental matrix Φ are linearly independent at $t\ \forall t \in I$

Conversely, let Φ be a solution matrix of the equation

$$\Phi'(t) = A(t)\Phi(\mathrm{t}). \quad (4.2)$$

Suppose that $det\ \Phi(t) \neq 0\ \forall t \in I$. Thus we have that *col* ($c = c_1,c_2,c_3,\ldots,c_n$) of Φ are linearly independent at every $t \in I$.

$\implies \Phi$ is a fundamental matrix.

Remark

A matrix of column vector may have a determinant identically zero on an interval I, although the vectors may be linearly independent.

Take, for example, Φ which is defined as

$$.\Phi(t) = \begin{pmatrix} t & t^2 \\ 0 & 0 \end{pmatrix} \text{ for any real interval.}$$

The above Theorem 4 just proved that this cannot occur for vectors which are solution of system (1.1).

Recommendation and Conclusion

Fundamental matrix function is an important function needed to find the solution and output function of controllable linear systems. Effort must be made, in research ventures, to determine easier ways or methods of getting such function. When that is done, search for solution and output function for controllable linear systems will be an easier one.

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