Stabilization of Controllable Systems

E. C. Mbah, S. E. Aniaku, P. U. Nwokoro, C. C. Asogwa
Department of Mathematics, University of Nigeria, Nsukka, Nigeria

Abstract. In this paper, the role played by functions that admit Lyapunov-like function and generalized Lyapunov-like function which also obey conditions such as (3.1) and (3.2) on the stabilization of controllable systems was studied. Examples of the application of such roles were also given.

Key words: Eigenvalues, Stability, Asymptotically, Exponentially, Uniformly

Introduction

The differential system of the form

$$\dot{x} = f(t) \text{ in } C' [0, \infty) \tag{1.1}$$

is stable about the origin if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|x_0| < 0$ implies that the solution $x(t)$ initiating at $x(0) = x_0$ satisfies $|x(t)| < \epsilon$ on $0 \leq t \leq \infty$.

We note that a necessary condition for stability is that $f(0) = 0$, which means that the origin a critical or rest or equilibrium point.

If $\frac{\partial f}{\partial x_i}(0)$ has all its eigenvalues with negative real parts, then the differential system is stable or even asymptotically stable about the origin.

The differential system (1.1) is asymptotically stable about the origin if for each $\epsilon > 0$, there exist a $\delta > 0$ such that $|x_0| < \delta$ implies that the solution $x(t)$ initiating at $x(0) = x_0$ satisfies $\underset{t \to \infty}{|x(t)|} < \epsilon$ $on\ 0 \leq t < \infty$ and $\lim_{t \to \infty} x(t) = 0$ (Wu, Yang, & Lin, 2013).

In addition to the above, if every solution of (1.1) in $\mathbb{R}^n$ is defined on $0 \leq t < \infty$ and tends towards the origin at $t \to \infty$, then (1.1) is said to be globally asymptotically stable (Wu, Yang, & Lin, 2013).

In this paper, we are interested in the control system in $\mathbb{R}^n$ of the form

$$\dot{x} = f(t, u) \text{ in } C^1 \text{ in } \mathbb{R}^{n+m} \tag{1.2}$$

with control restraint set $\Omega \in \mathbb{R}^m$

We also assume that there exists a scalar function $V(x)$ and a control m-vector $u(x)$ in C' in $\mathbb{R}^m$ such that

(a.) $V(x) \geq 0$ and $V(x) = 0$ *iff* $x = 0$.

(b.) $\lim_{|x| \to \infty} V(x) = +\infty$.

(c.) $u(x) \subset \Omega$.

(d.) $\frac{\partial V}{\partial \chi i} f_i(x, u(x)) < 0\ \forall\ i = 1, 2, \dots, n\ for\ x \neq 0$.

Under these conditions, the differential system

$$\dot{x} = f(x, u(x)) \tag{1.2}$$

is globally asymptotically stable at the origin and then stabilizable (Lee & Markus, 1976). Then, let us start with the following notations and definitions that will help us in this paper.

Notations:

$B(o,c)$ = Open ball centred at the origin with radius $c > 0$.

$\mathbb{R}^+$ = the set of all non- negative real numbers

$\mathbb{R}^n$ = the n - finite dimensional Euclidean space.

A^T = the transpose of a matrix A

$\lambda(A)$ = the set of all eigenvalues of A

ξ = the class of functions $\mu \in C^0(\mathbb{R}^+ \times \mathbb{R}^n)$ such that $\int_0^\infty \mu(t)dt < \infty$ and $\lim_{n\to\infty} \mu(t) = 0$.
C^0 = the class of continuous functions in $\mathbb{R}$
K^+ = the class of C^0 functions defined on $\mathbb{R}^+$
K_∞ = A positive definite increasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ which has $\lim_{n\to\infty} \rho(s) = +\infty$.

Definitions:

Definition 1. Lower Semi - Continuous functions (Canada, Drabek, & Fonda, 2006). A real-valued function $f(x)$ is lower semi- continuous at the point x_0, if for any small positive number $\epsilon, f(x)$ is always greater than $f(x) - \epsilon$ for all x in some neighborhood of x_0.
i.e. $f(x) > f(x_0) - \epsilon$

Definition 2. Upper Semi - Continuous functions (Canada, Drabek, & Fonda, 2006). A real-valued function $f(x)$ is upper semi-continuous at the point x_0, if for any small positive number $\epsilon, f(x)$ is always less than $f(x) + \epsilon$, for all x in some neighborhood of x_0.
i.e. $f(x) < f(x_0) + \epsilon$

Definition 3. (Eke & Aniaku, 2010)
The Autonomous linear control system in $\mathbb{R}^n$

$$\dot{x} = Ax + Bu \tag{1.4}$$

is said to be stabilizable if there exists a constant linear feedback control law $u = Dx$ such that

$$\dot{x} = Ax + BDx = (A + BD)x \tag{1.5}$$

is stable.
By definition 3, we mean that D is a real $m \times n$ matrix such that each eigenvalue of $A + BD$ has negative real part.
In other words, the changing of unstable eigenvalues (i.e. eigenvalues with non-negative real parts) to stable eigenvalues (i.e. eigenvalues with negative real parts). This is really what is called stabilization.

Definition 4. Control Lyapunov Function (CLF) (Aniaku & Jackreece, 2012). We say that $V : \mathbb{R}^+ \to \mathbb{R}^+$is CLF for the system (1.2) if $V(.)$ is lower semi-continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and there exists function $W : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ being upper semi-continuous and $a_1, a_2 \in K_\infty, \beta, \gamma \in K^+$ with $\int_0^\infty \beta(t)dt = +\infty, \mu \in \xi$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$being positive definite and lower continuous such that the following inequalities holds:
(1.) $a_1(|x|) \leq V(t,x) \leq a_2(y(t)|x|)$

$$\forall (t,x) \in \mathbb{R}^n \times \mathbb{R}^+ \tag{1.6}$$

(2.) $\inf_{x\in u}(t,x) \quad : f(t,x) \leq -w(t,x) + \beta(t)\mu(\int_0^\infty \beta(s)ds)$

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \setminus \{0\} \tag{1.7}$$

(3.)
$$W(t,x) \geq \beta(t).\rho\big(V(t,x)\big) \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \tag{1.8}$$

Definition 5. (Lee & Markus, 1976)
The system (1.2) is said to be globally asymptotically controllable to the origin if the following properties hold:
(1) For each $x_0 \in \mathbb{R}^n$, there exists an admissible control $u_0$such that the maximal caratheodory solution x starting from x_0 of

$$\dot{x} = f(x, u_0)$$

is defined for all $t \geq 0$ and satisfies $x(t) \to 0$ as $t \to \infty$.
(2.) For each > 0 there exists $c > 0$ such that for each $x_0 \in B(0,c)$, there is an admissible control u_0 as in 1 such that $x(t) \in B(0, \epsilon) \ \forall t \geq 0$.

Definition 6. Asymptotic Stabilization (Mohammed, 2010)
By asymptotic stabilization, we mean that the following two properties hold:
1. Stability of the origin of the closed - loop system
2. Convergence to the origin of all solutions.

Definition 7. Locally Asymptotically Stabilizable (Mohammed, 2010)
The system (1.2) is said to be locally asymptotically stabilizable if there exist a function $u = k(x), k: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, called feedback law, such that the implemented system

$$\dot{x}(t) = f(x, k(x)) \tag{1.9}$$

is locally asymptotically stable at the origin

Definition 8. Caratheodory Solution (Canada, Drabek, & Fonda, 2006)
By Caratheodory Solution of (1.2), we mean every solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\mathcal{T}, x(\mathcal{T}))d\mathcal{T} \tag{1.10}$$

Definition 9. Exponentially Stabilizable (Aniaku, 2017)
Control system (1.2) is exponentially stabilizable by the feedback control $u(t) = h(x(t))$, where $h(x): \mathbb{R}^n \longrightarrow \mathbb{R}^m$, if the closed-loop system

$$\dot{x}(t) = f(t, x(t)h(x(t))) \tag{1.11}$$

is exponentially stable.

Definition 10. Upper Right - hand Derivative of Function.
Let $V(t,x) : W \rightarrow D$ where $W = \mathbb{R}^+ \times D, D \subset \mathbb{R}^n$ be a given function, then the upper right-hand derivative of $V(t,x(t))$,*denoted* by $d^+V(t,x(t))$ is

$$d^+V\left(t, x(t)\right) = \limsup_{h \to 0+} \frac{V((t+h), x(t+h)) - V(t,(x(t))}{h} \tag{1.12}$$

Definition 11. Lipschitzian Function (Chidume, 1996)
A function $V(t,x(t)) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschitzian in x if there is a number $L > 0$ such that for all $t \in \mathbb{R}^+$

$$|V(t, x_1) - V(t, x_2)| \leq L|x_1 - x_2| \ \forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n \tag{1.13}$$

Definition 12. Lyapunov - Like Function (Mohammed, 2010)
A function $V(t,x) : W \rightarrow \mathbb{R}$ is called a Lyapunove-like function for (1.2) if $V(t,x)$ is continuously differentiable in $t \in \mathbb{R}^+$ and in $x \in D$, and there exist positive numbers $\lambda_1, \lambda_2, \lambda_3, k, p, q, r, \delta$ such that

1. $\lambda_1 \|x\|^p \leq V(t,x) \leq \lambda_2 \|x\|^q \forall (t,x) \in W$ (1.14)
2. $D_f V(t,x) \leq -\lambda_3 \|x\|^r + ke^{-\delta t}, \forall t > 0, x \in D \setminus \{0\}$ (1.15)

Definition 13. Exponentially stable (Aniaku, 2017)
The zero solution of system (1.2) is exponentially stable if any solution of (1.2) satisfies

$$\|x(t, x_0)\| \leq \beta(\|x_0\|, t_0)e^{-\delta(t-t_0)}, \forall t \geq t_0 \tag{1.16}$$

where $\beta(h,t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$is a non negative function increasing in $h \in \mathbb{R}^+$and δ is a positive constant.

Definition 14. Uniformly exponentially stable (Aniaku, 2017)
If the function $\beta(.)$ in definition 13 does not depend on t_0 then the zero solution is said to be uniformly exponentially stable.

Definition 15. Generalized Lyapunov - like Function.
A function $V(t,x) : W \rightarrow \mathbb{R}$ is called a generalized Lyapunov-like function for (1.2) if $V(t,x)$ is continuous in $t \in \mathbb{R}^+$and Lipschitzian in $x \in D$ and there exist positive numbers $\lambda_1, \lambda_2, \lambda_3, k, p, q, r, \delta$ such that;

1. $\lambda_1 \|x\|^p \leq V(t,x) \leq \lambda_2 \|x\|^q \forall (t,x) \in W$ (1.17)

$$2.\ D_f V\ (t,x) \leq -\ \lambda_3\ \|x\|^r + ke^{-\delta t}, \forall\ t\ \geq 0, x \in D \setminus \{0\} \tag{1.18}$$

Proposition and Remarks

Proposition 2.1

If there exists a continuous state feedback law rendering the origin of (1.2) a locally asymptotically stable equilibrium, then the map (x,u) → f(x,u) is open at zero.

Remark 2.1

It is known that if f(x) is continuous then the Cauchy problem

$$\begin{cases} \dot{x} = f(t, x(t)) \\ x(0) = x_0 \end{cases} \tag{2.1}$$

is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(z, x(z)\big)dz \tag{2.2}$$

Remark 2.2

It has been proved (Leigh, 1980) that the existence of a continuous control Lyapunov function (CLF) is a necessary and sufficient condition for the existence of a discontinuous feedback that stabilizes an autonomous control system. Moreover, as proved by Leigh (1980), the existence of a local Lipschitz LLF is equivalent to the existence of a stabilizing feedback. It was also pointed out (Wu, Yang, & Lin, 2013) that the existence of a time varying stabilizer of the form

$$u\ =\ k(t, x) \tag{2.3}$$

is equivalent to the existence of lower semi-continuous function.

Also all control systems which can be uniformly stabilized by means of continuous time-varying feedback can also be stabilized by means of smooth time-varying feedback. All control systems which can be uniformly stabilized by means of continuous time - varying feedback can also be stabilized by means of smooth time - varying feedback

It was also pointed out that the converse of the above statement is not true, for there exist system that can be stabilized by means of smooth time-varying feedback but cannot be uniformly stabilized by means of continuous time-varying feedback.

Main Result

Theorem 3.1:

The system (1.2) is uniformly exponentially stabilizable if it admits a Lyapunov-like function and the following two conditions hold for all (t,x) ∈ W :

$$1.\ \delta > \frac{\lambda_3}{|\lambda_2|^{\frac{r}{p}}} \tag{3.1}$$

and

$$2.\ \exists\gamma > 0\ such\ that\ V\ (t,x) - [V(t,x)]^{\frac{r}{q}} \leq \gamma e^{-\delta t} \tag{3.2}$$

Proof:

Consider any initial time t_0 and let x(t) be any solution of (1.2 with x(0) = x_0.

Let us set

$$Q(t,x) = V\ (t,x)e^{M(t-t_0)}, M = \frac{\lambda_3}{|\lambda_2|^{\frac{r}{q}}}$$

Then, $Q(t,x) = D_f V\ (t,x)e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}.$

Assuming the second condition of definition 12, for all t > $t_0, x \in D$ we have

$$\dot{Q}(t,x) \leq \big(-\lambda_3\ \|x\|^r + ke^{-\delta t}\big)e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0}$$

Then by the first condition of definition 15, we have

$\|x\|^q \geq \frac{V\ (t,x)}{\lambda_2}$ and equivalently - $\|x\|^r \leq -\left|\frac{V\ (t,x)}{\lambda_2}\right|^{\frac{r}{q}}$

So, we have

$$\dot{Q}(t,x) \leq \{-V\,(t,x)^{\frac{r}{q}}\,\frac{\lambda_3}{|\lambda_3|^{\frac{r}{q}}} + ke^{-\delta t}\}e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}$$

Since $\frac{\lambda_3}{|\lambda_2|^{\frac{r}{q}}} = M \quad \forall\, t \geq 0$, we have

$$\dot{Q}(t,x) \leq M\{V(t,x) - V(t,x)^{\frac{r}{q}}e^{M(t-t_0)} + Ke^{M(t-t_0)}$$

Making use of (3.2) we have

$$\dot{Q}(t,x) \leq (M\gamma + K)e^{(M-\delta)(t-t_0)}$$

Integrating both sides from t_0 to t, we get

$$Q(t,x) - Q\,(t_0 x_0) \leq \int_{t_0}^{t}(M\gamma + k)e^{(M-\delta)(s-t_0)}ds$$

$$= (M\gamma + k)\frac{1}{M-\delta}\{e^{(M-\delta)(t-t_0)} - 1\}$$

Setting $\delta_1 = -(M-\delta)$, we see that $\delta > 0$ and so, we get

$$Q(t,x) \leq Q\,(t_0,x_0) + \frac{M\gamma + k}{\delta_1} - \frac{M\gamma + k}{\delta_1}e^{(M-\delta)(t-t_0)}$$

$$= Q(t_0,x_0) + \frac{M\gamma + k}{\delta_1}$$

Since $Q(t_0,x_0) = V(t_0,x_0) \leq \lambda_2\,\|x_0\|^q$, we get

$$Q(t,x) \leq \lambda_2\,\|x_0\|^q + \frac{M\gamma + k}{\delta_1}$$

Setting $\lambda_2\,\|x_0\|^q + \frac{M\gamma+k}{\delta_1} = \beta(\|x_0\|) > 0$

we have

$$Q\,(t,x) \leq \beta(\|x_0\|)\forall t \geq t_0 \tag{3.3}$$

On the other hand, from (1.17), it follows that

$\lambda_1\,\|x\|^p \leq V\,(t,x)$

$$\Longrightarrow \|x\|^p \leq \{\frac{V\,(t,x)}{\lambda_1}\}^{\frac{1}{p}} \tag{3.4}$$

Setting $V(t,x) = Q(t,x)e^{-M(t-t_0)}$ into (3.4)

we get

$$\|x\| \leq \{\frac{Q(t,x)e^{-M(t-t_0)}}{\lambda_1}\}^{\frac{1}{p}} \tag{3.5}$$

From (3.4) and (3.5) we have

$$\|x\,(t)\| \leq \{\frac{\beta(\|x_0\|)e^{-M(t-t_0)}}{\lambda_1}\}^{\frac{1}{p}}$$

$$= \{\frac{\beta(\|x_0\|)}{\lambda_1}\}^{\frac{1}{p}}\,e^{\frac{-M}{P}(t-t_0)}, \forall t \geq t_0 \tag{3.6}$$

This shows that (2.1) is uniformly exponentially stable and stabilizable.

Theorem 3.2

The system (2.1) is exponentially stable, it admits a generalized Lyapunov - like function and these two conditions hold for all $(t,x) \in W$:

$$(1.)\;\delta > \inf_{t\in\mathbb{R}^+}\frac{\lambda_3(t)}{|\lambda_2(t)|^{\frac{r}{q}}} > 0 \tag{3.7}$$

and

$$(2.)\ \exists\, \gamma > 0 \text{ such that } V(t,x) - [V(t,x)]^{\frac{r}{q}} \leq \gamma e^{-\delta t} \tag{3.8}$$

Proof:

Let us consider the function

$$Q(t,x(t)) = V(t,x)e^{M(t-t_0)} \tag{3.9}$$

Where $M = \inf_{t\in\mathbb{R}^+} \frac{\lambda_3(t)}{|\lambda_2(t)|^{\frac{r}{q}}}$

We see that $\delta \geq M > 0$

$$D_f^+ Q(t,x_0) = D_f^+ Q(t,x)e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}$$

Proceeding as above, we have

$$D_f^+ Q(t,x) \leq \left(-\lambda_3 \|x\|^r + Ke^{-\delta t}\right)e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}$$

Since $\lambda_2(t) > 0 \forall t \in \mathbb{R}^+$, and by (1.17) we have,

$$\|x\|^q \geq \frac{V(t,x)}{\lambda_2}$$

Equivalently,

$$-\|x\|^r \leq -[\frac{V(t,x)}{\lambda_2}]^{\frac{r}{q}}$$

So, we have $D_f^+ Q(t,x) \leq \{-V(t,x)^{\frac{r}{q}} \frac{\lambda_3}{|\lambda_2|^{\frac{r}{q}}} + Ke^{-\delta t}\}e^{M(t-t_0)} + MV(t,x)e^{M(t-t_0)}$

Since $\frac{\lambda_3}{|\lambda_2|^{\frac{r}{q}}} \geq M \geq 0$, , by (3.7) and by the condition (3.8), we get

$$\begin{aligned} D_f^+ Q(t,x) &\leq M\left\{V(x,t) - [V(t,x)]^{\frac{r}{q}}\right\}e^{M(t-t_0)} + Ke^{-\delta t}e^{M(t-t_0)} \\ &\leq M\gamma e^{-\delta t}e^{M(t-t_0)} + Ke^{-\delta t}e^{M(t-t_0)} \\ &= (M\gamma + K)e^{-\delta(t-t_0)}e^{M(t-t_0)} \end{aligned}$$

So,

$$D_f^+ Q(t,x) \leq (M\gamma + K)e^{(M-\delta)(t-t_0)} \tag{3.10}$$

Integrating both sides of (3.10) from t_0 to t, we have

$$\begin{aligned} Q(t,x) - Q(t_0,x_0) &\leq \int_{t_0}^{t} (M\gamma + K)e^{(M-\delta)(s-t_0)}ds \\ &= (M\gamma + K)\frac{1}{M-\delta}\{e^{(M-\delta)(t-t_0)} - 1\} \end{aligned}$$

Setting $\delta_1 = -(M - \delta)$, then we have $\delta_1 > 0$, and using the condition (3.7), we have

$$\begin{aligned} Q(t,x) &\leq Q(t_0,x_0) + \frac{M\gamma + k}{\delta_1} - \frac{M\gamma + k}{\delta_1}e^{(M-\delta)(t-t_0)} \\ &\leq Q(t_0,x_0) + \frac{M\gamma + k}{\delta_1} \end{aligned}$$

From (3.8) and (3.9), we get that

$Q(t_0,x_0) \leq V(t_0,x_0) \leq \lambda_2 \|x_0\|^q$ and

$Q(t,x) \leq \lambda_2 \|x_0\|^q + \frac{M\gamma + k}{\delta_1}$

Set $\lambda_2 \|x_0\|^q + \frac{M\gamma + k}{\delta_1} = \beta(\|x_0\|, t_0) > 0$ to get

$$Q(t,x) \leq \beta(\|x_0\|, t_0), \forall t \leq t_0 \tag{3.11}$$

Also, from condition (1.17), we have

$\lambda_1 \|x\|^q \leq V(t,x)$.

$$\Rightarrow \|x\| \leq \left\{\frac{V(t,x)}{\lambda_1}\right\}^{\frac{1}{p}} \tag{3.12}$$

Substituting and using (3.9) into (3.12), we get

$$\|x\| \leq \left\{\frac{Q(t,x)}{\lambda_1} e^{-M(t-t_0}\right\}^{\frac{1}{p}} \tag{3.13}$$

Combining (3.11) into (3.13), we get

$$\|x\| \leq \left\{\frac{\beta(\|x_0\|,t_0)}{\lambda_1}\right\}^{\frac{1}{p}} = \left\{\frac{\beta(\|x_0\|,t_0)}{\lambda_1}\right\}^{\frac{1}{p}} e^{\frac{-M}{p}(t-t_0),} \ \forall t \geq t_0 \tag{3.14}$$

This then shows that (2.1) is exponentially stable and then stabilizable.

Examples

Example 4.1. Consider a non-linear differential equation

$$\dot{x} = -\frac{1}{4}x^{\frac{3}{5}} + xe^{-2t}, t \geq 0 \tag{4.1}$$

Taking Lyapunov-like function

$V(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^+$ as $V(t,x) = x^6$, where $D = \{x : |x| \leq 1\}$

We note that $x^7 \leq V(t,x) \leq |x|^6 \ \forall x \in D$.

Then, condition (1.17) holds with $\lambda_1 = \lambda_2 = 1, p = 7, q = 6$.

Also,

$$\dot{V}(t,x) = 6x^5\dot{x}$$
$$= 6x^5\left(-\frac{1}{4}x^{\frac{3}{5}} + xe^{-2t}\right)$$
$$= -\frac{3}{2}x^{\frac{28}{5}} + 6x^6e^{-2t}$$

So,

$$\dot{V}(t,x) \leq -\frac{3}{2}x^{\frac{28}{5}} + 6x^6e^{-2t} \quad \forall x \in D$$

We see that conditions (3.1) and (3.2) of the main Theorem 1 are also satisfied for

$$V(t,x) - [V(t,x)]^{\frac{r}{q}} = x^6 - x^{\frac{28}{5}} = x^{\frac{28}{5}}\left(x^{\frac{2}{5}} - 1\right) \leq 0 \leq e^{-2t} \quad \forall x \in D$$

Therefore, (4.1) is exponentially stable and so is stabilizable.

Example 4.2. Consider a non-linear differential equation

$$\dot{x} = -\frac{1}{2}x^{\frac{2}{3}} + xe^{-3t}, t \geq 0 \tag{4.2}$$

Taking Lyapunov-like function

$V(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^+$ as $V(t,x) = x^4$, where $D = \{x : |x| \leq 1\}$

We note that $x^7 \leq V(t,x) \leq |x|^5 \leq V(t,x) \leq |x|^4 \ \forall x \in D$.

Then, condition (1.17) is satisfied with $\lambda_1 = \lambda_2 = 1, p = 5, q = 4$.

Also,

$$\dot{V}(t,x) = 4x^3\dot{x} = 4x^3\left[-\frac{1}{2}x^{\frac{2}{3}} + xe^{-3t}\right]$$
$$= -2x^3x^{\frac{2}{3}} + 4x^4e^{-3t}$$
$$= -2x^{\frac{11}{3}} + 4x^4e^{-3t}$$

So,

$$\dot{V}(t,x) \leq -2x^{\frac{11}{3}} + 4x^4e^{-3t}$$

We see also that conditions (1.17) and (1.18) of definition 15 are satisfied with $\lambda_3 = 2$, $K = 4, \delta = 3$ and. $r = \frac{11}{3}$

Moreover, conditions (3.1) and (3.2) of Theorem 1 are also satisfied. for

$$V(t,x) - [V(t,x)]^{\frac{r}{q}} = x^4 - (x^4)^{\frac{11}{3} \div 4}$$

$$= x^4 - (x^4)^{\frac{11}{12}} = x^4 - x^{\frac{11}{3}}$$
$$= x^{\frac{11}{3}}[x^{\frac{1}{3}} - 1] \leq 0 \leq e^{-3t} \forall x \in D$$

Therefore, (4.2) is exponentially stable.

Conclusion

From these, we have shown that functions which admit Lyapunove-Like functions and which also obey condition (3.1) and (3.2) are uniformly exponentially stabilizable.

Also, it was shown that functions which admit generalized Lyapunove-Like functions and obey two conditions (3.7) and (3.8) are exponentially stable.

References

Aniaku, S. E. & Jackreece, P. C. (2012). Stability for Non-Linear Systems that admit some Lyapunov-Like functions. *Journal of the Nigerian Association of Mathematical Physics, 21*, 309-314.

Aniaku, S. E. (2017). Exponential Stability and Robustness of Control Systems. *International Journal of Engineering Sciences Invention, 6*(9), 47-49.

Canada, A., Drabek, P., & Fonda, A. (2006). Hand Book of Differential Equations. *Ordinary Differential Equations, III*.

Chidume, C. E. (1996). *Applicable Functional Analysis. Fundamental Theorems with Applications*. Trieste, Italy.

Clarke, F. H. & Stem, R. J. (2003). State Constrained Feedback Stabilization. *SIAM J. Control Optim., 42*(2), 422-441.

Eke, A. N. & Aniaku, S. E. (2010). Stabilizability for Systems With Non-Linear Pertubations. *Global Journal of Mathematical Sciences, 9*(2), 95-101.

Lee, E. B. & Markus, L. (1976). *Functions of Optimal Control Theory*. New York: John Wiley and Sons Inc.

Leigh, J. R. (1980). *Functional Analysis and Linear Control Theory*. New York: Academic Press.

Mohammed, O. (2010). Exponential and Weak Stabilization of Constrained Bilinear Systems. *SIAM J. Control Optim., 48*(6), 3962-3974.

Wu, M., Yang, Z., & Lin, W. (2013). Exact asymptotic stability analysis and region-of-attraction estimation for nonlinear systems. *Abstract and Applied Analysis*, Article ID 146137.