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An Another Approach for Buckling Calculus of Dental Implant as a Bar on Elastic Environment by Transfer-Matrix Method (TMM)

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Abstract. Applications in orthodontics of mathematical approach for dental implants are in full development, due to the rapid evolution in computerization of all medicine branches. In this paper we present a work in which we consider the dental implant as a bar on elastic environment, one-end of bar embedded and other end as a guided joint, with an axial compression force *P* at the joint end. That is an original idea to assimilate the dental implant as a bar embedded in an elastic environment. The bone is assimilated as an elastic environment. We used the Transfer-Matrix Method, because is very easy to program and so, we can obtain instant results with quick application in practice, even in situ. This work with this approach will serve for future research about dental implants, for optimization forms of implants.

Key words: dental implant, elastic environment, state vector, Transfer-Matrix

Introduction

Applications in orthodontics of mathematical approach for dental implants are in full development, due to rapid evolution in computerization of all medicine branches. For our future research, this approach is fundamental for studies about titanium and its alloys of medicine domains, for prosthesis, in occurrence, in orthodontics, for dental implants. In this paper, we present an original relatively simple analytical calculus for critical buckling force. That is an original idea to assimilate the dental implant as a bar embedded in an elastic environment. The algorithm is based on the mathematical approach offered by Dirac's and Heaviside's functions and operators, with directly applications for dental implants. Approach for this problem of buckling bars on elastic environment is analyzed by TMM, providing a validation basis for future experimental tests of some of bio-composite materials. After, we want to validate the algorithm by Finite Element Method (FEM).

Literature Review

In the scientific world, the concern of researchers on dental implants is very wide, as can be seen from the numerous publications in the field, of which, in the following, we will present some of them, both from the technical and from the medical field. A study is presented about a high order finite strip transfer matrix method for buckling analysis of single-branched cross-section thin-walled members (Bin et al., 2019). We have a study about osseointegration of titanium, titanium alloy and zirconia dental implants (Bosshardt, Chappuis & Buser, 2017). It can be seen the influence of surface characteristics on bone integration of titanium implants on pigs (Buser, 1991). We also have a recent development in beta titanium alloys for biomedical applications (Chen, 2020). They have developed the basics of Transfer-Matrix Method and its applications (Gery & Calgaro, 1987). We see studies about the role of primary stability for successful osseointegration of dental implants and the factors of influence and its evaluation too (Javed et al., 2013). A biomechanical analysis of alveolar bone stress around implants with different thread designs and pitches in the mandibular molar area is presented (Lan et al., 2012). An application of TMM for buckling of Piles in Layered Soils is given too (Lu, Tao & Qijian, 2021). We have a study about titanium-based biomaterials for preventing stress shielding between implant devices and bone (Niinom & Nakai, 2011). A study about the influence of implant taper on the primary and secondary stability of osseointegrated titanium implants is given too (O'Sullivan, Sennerby & Meredith, 2004). They are presented studies about mechanical reliability, fatigue strength and survival analysis of new polycrystalline translucent zirconia ceramics for monolithic restorations (Pereira et al., 2018). A review of influence of thread geometry on implant osseointegration under immediate loading is presented too (Ryu et al., 2014). They have presented us the importance of TMM for elastic-buckling analysis of compression bar (Sun & Li, 2011). They give us a study of bending beam on elastic environment by TMM (Suciu, 2012) and we have presented a study about buckling bio-composite sandwich bars too (Suciu, 2013). Also, the bases of Strength of Materials are found (Suciu, 2009). We have presented our first study about buckling calculus of straight bars on elastic environment by TMM for dental implants (Tripa et al., 2018). They study the effect of implant thread geometry on secondary stability, bone density, and bone-to-implant in a biomechanical and histological analysis contact (Trisi et al., 2015). A review of the surface modifications of titanium alloys for biomedical applications is presented too (Uporabo, 2017). It is given relevant aspects in the surface properties in titanium dental implants for the cellular viability (Velasco-Ortega et al., 2016). We have a synthesis of formulas for stress and strain (Warren, 1989). Fabrication and properties of functionally graded dental implant are presented too (Watari et al., 1997). It is given tapered implants in dentistry with revitalizing concepts with technology (Wilson et al., 2016). Interesting research in presented about the production and characterization of a bone-like porous Ti/Ti-Hydroxyapatite functionally graded material (Yilmaz et al., 2020).

Materials and Methods

Titanium and its alloys are most widely used due to mechanical properties like the bone tissue: a good hardness and a good corrosion resistance. Risks of buckling are higher for straight bars compressed with an axial force. It is very important knowing the critical buckling force. We consider the dental implant as a bar on elastic environment, one-end of bar embedded and other end as a guided joint, with an axial compression force *P* at the joint end. The bone is assimilated as an elastic environment.

Analytical Calculus for Buckling Straight Bars with Axial Compression Force on Rigid Environment

It is considered a dental implant (Figure 1).

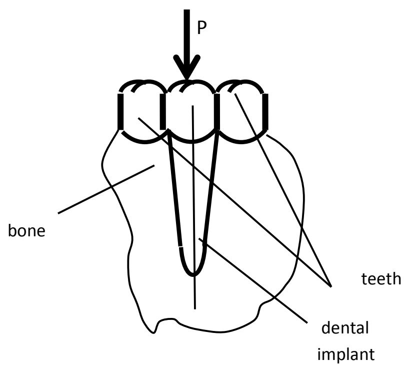

Figure 1. A dental implant

Dental implant can be assimilated with a straight bar embedded in the lower part and with a guided joint in the upper part, where a vertical axial force *P* acts (Figure 2).

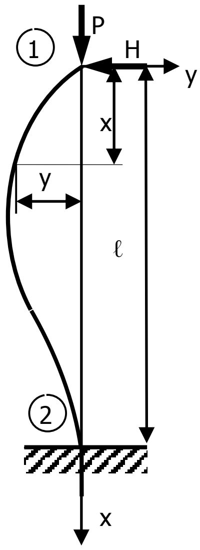

Figure 2. A straight bar embedded in the lower part and with a guided joint in the upper part, where a vertical axial force *P* acts

The implant deforms, limited by the neighboring teeth, from his right and from his left. The bone on which the dental implant is placed is an elastic environment.

Average Deformed Fiber for a Bar, which Rests on a Rigid Environment

To begin, we will study a straight bar on a rigid medium, embedded at one end and with a guided joint at the other end, stressed with an axial compression force *P* acting on the articulated end (Figure 2).

Bar has the length *l*, with constant moment of inertia *I* over the entire length. The compression force *P* acts in direction of axis of bar in end *1*, which tend to compress the bar.

In section *x*, the bending moment *M(x)* is (1) (Suciu, 2009):

$$M(x) = P \cdot y - H \cdot x \tag{1}$$

The differential equation of the deformed fiber is (2):

$$EI\frac{d^2y}{dx^2} = -M(x) \tag{2}$$

where *E* is Young's modulus. With (1), we can write (3):

$$EI\frac{d^2y}{dx^2} = -P \cdot y + H \cdot x \tag{3}$$

For medium deformed fiber, the differential equation (3) is derived twice, and we obtained successively (4), (5), or (6), (Gery & Calgaro, 1987):

$$EI\frac{d^3y}{dx^3} = -P\frac{dy}{dx} + H \tag{4}$$

$$EI\frac{d^4y}{dx^4} = -P\frac{d^2y}{dx^2} \tag{5}$$

or:

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0 \tag{6}$$

A homogeneous equation of degree 4, (6), was thus obtained.

The following consecrated notation (7) is made:

$$\alpha^2 = \frac{P}{EI} \tag{7}$$

With (7), the equation (6) becomes (8):

$$\frac{d^4y}{dx^4} + \alpha^2 \frac{d^2y}{dx^2} = 0 \tag{8}$$

Characteristic equation for (8) is (9):

$$r^4 + \alpha^2 r^2 = 0 \tag{9}$$

with general solution (10):

$$y(x) = C_1 \cos\alpha x + C_2 \sin\alpha x + C_3 x + C_4 \tag{10}$$

where C_i, $i=1,4$ are constants of integration.

Expression (10) is successively derived 4 times and is obtained relations (11):

$$\begin{cases} y(x) = C_1 \cos\alpha x + C_2 \sin\alpha x + C_3 x + C_4 \\ \frac{dy}{dx} = -C_1 \alpha \sin\alpha x + C_2 \alpha \cos\alpha x + C_3 \\ \frac{d^2y}{dx^2} = -C_1 \alpha^2 \cos\alpha x - C_2 \alpha^2 \sin\alpha x \\ \frac{d^3y}{dx^3} = C_1 \alpha^3 \sin\alpha x - C_2 \alpha^3 \cos\alpha x \\ \frac{d^4y}{dx^4} = C_1 \alpha^4 \cos\alpha x + C_2 \alpha^4 \sin\alpha x \end{cases} \tag{11}$$

The State Vector Associated with a Section *x* of the Bar which Rests on a Rigid Environment

The foundations for TMM are given (Gery & Calgaro, 1987). The state vector for section *x* can be written as a column vector with 4 elements, as follows (12):

$$\{V\}_x = \{V(x)\} = \{y(x), \omega(x), T(x), M(x)\}^{-1} \tag{12}$$

where:

- $\{V(x)\}_x$ is the state vector for section *x*
- *y(x)* is the horizontal linear displacement of the section
- *ω(x)* is the angular deformation near the *x* section
- *T(x)* is the cutting force in section *x*
- *M(x)* is the bending moment in section *x*.

It was considered an orthogonal biaxial reference system with the origin at point *1*, with the vertical axis *x*, along symmetry axis of bar and with the horizontal axis *y*. A state vector, with index *0*, (13), is associated in the origin, the section *0*, for *x=0*:

$$\{V\}_0 = \{y_0, \omega_0, T_0, M_0\}^{-1} = \{V(0)\} = \{y(0), \omega(0), T(0), M(0)\}^{-1} \tag{13}$$

For *x=l*, at the other end of bar is associated the state vector (14):

$$\{V\}_l = \{V(l)\} = \{y(l), \omega(l), T(l), M(l)\}^{-1} \tag{14}$$

The Transfer-Matrix Associated of a Straight Bar

We can write the connection relation (15) between the origin section *0* and the section *x*, with the Transfer-Matrix $[Tm]_x$:

$$\{V\}_x = [Tm]_x \{V\}_0 + [Fe]_x \{Ve\}_x \tag{15}$$

where:

- $\{V\}_x$ is the state vector corresponding to the section *x*

- *[Fe]x* is the matrix corresponding to the external loads in section *x*
- *{Ve}x* is the state vector corresponding to the external loads in section ***x*.**

With conditions in the origin, for *x=0*, the integration constants C_1, C_2, C_3, C_4 are obtained, as function of state vector in the origin (16):

$$\begin{cases} C_1 = -\dfrac{M_0}{EI\alpha^2} \\ C_2 = -\dfrac{T_0}{EI\alpha^3} \\ C_3 = \omega_0 + \dfrac{T_0}{EI\alpha^2} \\ C_4 = y_0 + \dfrac{M_0}{EI\alpha^2} \end{cases} \tag{16}$$

For homogeneous differential equation, solution becomes (17):

$$\begin{cases} y(x) = y_0 + x\omega_0 + \dfrac{\alpha x - \sin\alpha x}{EI\alpha^3} T_0 + \dfrac{1-\cos\alpha x}{EI\alpha^2} M_0 \\ \omega(x) = \omega_0 + \dfrac{1-\cos\alpha x}{EI\alpha^2} T_0 + \dfrac{\sin\alpha x}{EI\alpha} M_0 \\ T(x) = \dfrac{\cos\alpha x}{EI} T_0 - \dfrac{\alpha\sin\alpha x}{EI} M_0 \\ M(x) = \dfrac{\sin\alpha x}{EI\alpha} T_0 + \dfrac{\cos\alpha x}{EI} M_0 \end{cases} \tag{17}$$

With the four expressions from (17), we can write the matrix relation (18):

$$\{V(x)\} = \begin{Bmatrix} y(x) \\ \omega(x) \\ T(x) \\ M(x) \end{Bmatrix} = \begin{bmatrix} 1 & x & \dfrac{\alpha x - \sin\alpha x}{EI\alpha^3} & \dfrac{1-\cos\alpha x}{EI\alpha^2} \\ 0 & 1 & \dfrac{1-\cos\alpha x}{EI\alpha^2} & \dfrac{\sin\alpha x}{EI\alpha} \\ 0 & 0 & \dfrac{\cos\alpha x}{EI} & -\dfrac{\alpha\sin\alpha x}{EI} \\ 0 & 0 & \dfrac{\sin\alpha x}{EI\alpha} & \dfrac{\cos\alpha x}{EI} \end{bmatrix} \begin{Bmatrix} y_0 \\ \omega_0 \\ T_0 \\ M_0 \end{Bmatrix} \tag{18}$$

For the section from the bottom, for *x=l*, relation (18) becomes (19):

$$\{V(l)\} = \begin{Bmatrix} y(l) \\ \omega(l) \\ T(l) \\ M(l) \end{Bmatrix} = \begin{bmatrix} 1 & l & \dfrac{\alpha l - \sin\alpha l}{EI\alpha^3} & \dfrac{1-\cos\alpha l}{EI\alpha^2} \\ 0 & 1 & \dfrac{1-\cos\alpha l}{EI\alpha^2} & \dfrac{\sin\alpha l}{EI\alpha} \\ 0 & 0 & \dfrac{\cos\alpha l}{EI} & -\dfrac{\alpha\sin\alpha l}{EI} \\ 0 & 0 & \dfrac{\sin\alpha l}{EI\alpha} & \dfrac{\cos\alpha l}{EI} \end{bmatrix} \begin{Bmatrix} y_0 \\ \omega_0 \\ T_0 \\ M_0 \end{Bmatrix} \tag{19}$$

The conditions for the two extremes are (20):

$$\begin{cases} y_0 = 0 \\ M_0 = 0 \\ y(l) = 0 \\ \omega(l) = 0 \end{cases} \tag{20}$$

With (20), relation (19) becomes (21):

$$\begin{Bmatrix} 0 \\ 0 \\ T(l) \\ M(l) \end{Bmatrix} = \begin{bmatrix} 1 & l & \frac{\alpha l - \sin\alpha l}{EI\alpha^3} & \frac{1-\cos\alpha l}{EI\alpha^2} \\ 0 & 1 & \frac{1-\cos\alpha l}{EI\alpha^2} & \frac{\sin\alpha l}{EI\alpha} \\ 0 & 0 & \frac{\cos\alpha l}{EI} & -\frac{\alpha\sin\alpha l}{EI} \\ 0 & 0 & \frac{\sin\alpha l}{EI\alpha} & \frac{\cos\alpha l}{EI} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_0 \\ T_0 \\ 0 \end{Bmatrix} \tag{21}$$

The following linear system of 4 equations with 4 unknowns is thus obtained (22):

$$\begin{cases} 0 = l\omega_0 - \frac{\alpha l - \sin\alpha l}{EI\alpha^3} T_0 \\ 0 = \omega_0 + \frac{1-\cos\alpha l}{EI\alpha^3} T_0 \\ T(l) = \frac{\cos\alpha l}{EI} T_0 \\ M(l) = \frac{\sin\alpha l}{EI\alpha} T_0 \end{cases} \tag{22}$$

From the second equation of (22), we have (23):

$$\omega_0 = -\frac{1-\cos\alpha l}{EI\alpha^3} T_0 \tag{23}$$

(23) is replaced in (22) and the trigonometric equation is obtained (24):

$$tg\alpha l = \alpha l \tag{24}$$

with solution (25):

$$\alpha^2 l^2 = 2\pi^2 \tag{25}$$

where from (26):

$$\alpha^2 = \frac{2\pi^2}{l^2} \tag{26}$$

and with (7), we have (27):

$$\alpha^2 = \frac{2\pi^2}{l^2} = \frac{P}{EI} \tag{27}$$

Now, we can calculate the critical buckling force, the same as in (Suciu, 2009).

Calculus for a Dental Implant as a Buckling Straight Bar with Axial Compression Force on Elastic Environment

The dental implant is assimilated with a bar embedded in an elastic medium at the bottom and with a support on a guide at the top. The dental implant is assimilated with a bar embedded in an elastic medium at the bottom and with a support on a guide at the top, the guidance being provided by the neighboring teeth, from the left and right of the implant. The guidance prevents lateral displacement of the implant (Figure 3).

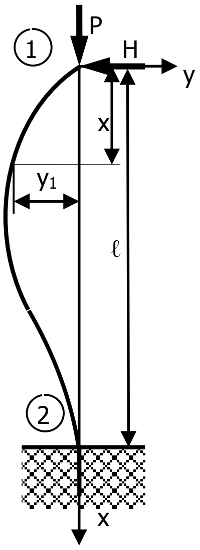

Figure 3. A buckling straight bar with axial compression force on elastic environment

We try to derive the transfer matrix for a bar that is on an elastic environment. Elastic medium can be perfectly uniform and homogeneous, and we must know the function for coefficient of elasticity of elastic environment *k(x)*. It was considered a working hypothesis that the arrows at the two ends of the bar are initially *0*.

Average Deformed Fiber for a Bar, which Rests on Elastic Environment

For the deformed average fiber $y_1(x)$ have the differential equation for a bar which is embedded in an elastic environment (28):

$$EI\frac{d^4y_1}{dx^4}+P\frac{d^2y_1}{dx^2}+ky_1=0 \tag{28}$$

The following notations, (29) and (30), are made:

$$2b=\frac{P}{EI} \tag{29}$$

$$\varepsilon=\frac{k}{EI} \tag{30}$$

For the homogeneous linear differential equation of order 4 (28), we have the characteristic equation (31):

$$r_1^4+2b\cdot r_1^2+\varepsilon=0 \tag{31}$$

The determinant for equation (31) is (32):

$$\Delta=\sqrt{b^2-\varepsilon} \tag{32}$$

In this case, we have the following possible situations:

- if $b^2<\varepsilon$ – does not exist solution
- if $b^2>\varepsilon$ and after [5], we can write (33):

$$\begin{cases}\delta_1=b+\sqrt{\Delta}\\ \delta_2=b-\sqrt{\Delta}\end{cases} \tag{33}$$

For equation (31), the general solution is (34):

$$y_1(x)=D_1\cos\delta_1x+D_2\sin\delta_1x+D_3\cos\delta_2x+D_4\sin\delta_2x \tag{34}$$

By deriving the relation (34) times 4, the expressions (35) are obtained:

$$\begin{cases} y_1(x) = D_1\cos\delta_1 x + D_2\sin\delta_1 x + D_3\cos\delta_2 x + D_4\sin\delta_2 x \\ \dfrac{dy_1}{dx} = -D_1\delta_1\sin\delta_1 x + D_2\delta_1\cos\delta_1 x - D_3\delta_2\sin\delta_2 x + D_4\delta_2\cos\delta_2 x \\ \dfrac{d^2y_1}{dx^2} = -D_1\delta_1^2\cos\delta_1 x - D_2\delta_1^2\sin\delta_1 x - D_3\delta_2^2\cos\delta_2 x - D_4\delta_2^2\sin\delta_2 x \\ \dfrac{d^3y_1}{dx^3} = D_1\delta_1^3\sin\delta_1 x - D_2\delta_1^3\cos\delta_1 x + D_3\delta_2^3\sin\delta_2 x - D_4\delta_2^3\cos\delta_2 x \\ \dfrac{d^4y_1}{dx^4} = D_1\delta_1^4\cos\delta_1 x + D_2\delta_1^4\sin\delta_1 x + D_3\delta_2^4\cos\delta_2 x + D_4\delta_2^4\sin\delta_2 x \end{cases} \tag{35}$$

With conditions in origin, i.e., in point *1* (Figure 3) for $x=0$, and doing the calculations, the integration constants D_1, D_2, D_3, D_4, are obtained (36):

$$\begin{cases} D_1 = \dfrac{\delta_2^2}{(\delta_2^2-\delta_1^2)EI} y_0' + \dfrac{1}{(\delta_2^2-\delta_1^2)EI} M_0' \\ D_2 = \dfrac{\delta_2^2}{\delta_1(\delta_2^2-\delta_1^2)EI} \omega_0' + \dfrac{1}{\delta_1(\delta_2^2-\delta_1^2)EI} T_0' \\ D_3 = -\dfrac{\delta_1^2}{(\delta_2^2-\delta_1^2)EI} y_0' - \dfrac{1}{(\delta_2^2-\delta_1^2)EI} M_0' \\ D_4 = -\dfrac{\delta_1^2}{\delta_2(\delta_2^2-\delta_1^2)EI} \omega_0' - \dfrac{1}{\delta_2(\delta_2^2-\delta_1^2)EI} T_0' \end{cases} \tag{36}$$

With (36), the general solution becomes (37):

$$\begin{aligned} y_1(x) = &\left(\frac{\delta_2^2}{(\delta_2^2-\delta_1^2)EI} y_0' + \frac{1}{(\delta_2^2-\delta_1^2)EI} M_0'\right)\cos\delta_1 x + \left(\frac{\delta_2^2}{\delta_1(\delta_2^2-\delta_1^2)EI}\omega_0' + \frac{1}{\delta_1(\delta_2^2-\delta_1^2)EI} T_0'\right)\sin\delta_1 x + \\ &+\left(-\frac{\delta_1^2}{(\delta_2^2-\delta_1^2)EI} y_0' - \frac{1}{(\delta_2^2-\delta_1^2)EI} M_0'\right)\cos\delta_2 x + \left(-\frac{\delta_1^2}{\delta_2(\delta_2^2-\delta_1^2)EI}\omega_0' - \frac{1}{\delta_2(\delta_2^2-\delta_1^2)EI} T_0'\right)\sin\delta_2 x \end{aligned} \tag{37}$$

or:

$$\begin{aligned} y_1(x) = &\left(\frac{\delta_2^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI} y_0' + \frac{\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI} M_0'\right) + \left(\frac{\delta_2^2\sin\delta_1 x}{\delta_1(\delta_2^2-\delta_1^2)EI}\omega_0' + \frac{\sin\delta_1 x}{\delta_1(\delta_2^2-\delta_1^2)EI} T_0'\right) + \\ &+\left(-\frac{\delta_1^2\cos\delta_2 x}{(\delta_2^2-\delta_1^2)EI} y_0' - \frac{\cos\delta_2 x}{(\delta_2^2-\delta_1^2)EI} M_0'\right) + \left(-\frac{\delta_1^2\sin\delta_2 x}{\delta_2(\delta_2^2-\delta_1^2)EI}\omega_0' - \frac{\sin\delta_2 x}{\delta_2(\delta_2^2-\delta_1^2)EI} T_0'\right) \end{aligned} \tag{38}$$

and grouping the components of expression (38), we get (39) for $y_1(x)$:

$$\begin{aligned} y_1(x) = &\frac{\delta_2^2\cos\delta_1 x - \delta_1^2\cos\delta_2 x}{(\delta_2^2-\delta_1^2)EI} y_0' + \frac{\delta_2^3\sin\delta_1 x - \delta_1^3\sin\delta_2 x}{\delta_1\delta_2(\delta_2^2-\delta_1^2)EI}\omega_0' + \\ &+\frac{\delta_2\sin\delta_1 x - \delta_1\sin\delta_2 x}{\delta_1\delta_2(\delta_2^2-\delta_1^2)EI} T_0' + \frac{\cos\delta_1 x - \cos\delta_2 x}{(\delta_2^2-\delta_1^2)EI} M_0' \end{aligned} \tag{39}$$

Simplified, it can be written (40):

$$y_1(x) = a_{11}y_0' + a_{12}\omega_0' + a_{13}T_0' + a_{14}M_0' \tag{40}$$

with notations (41):

$$\begin{cases} a_{11} = \dfrac{\delta_2^2 \cos\delta_1 x - \delta_1^2 \cos\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} \\ a_{12} = \dfrac{\delta_2^3 \sin\delta_1 x - \delta_1^3 \sin\delta_2 x}{\delta_1\delta_2\left(\delta_2^2 - \delta_1^2\right)EI} \\ a_{13} = \dfrac{\delta_2 \sin\delta_1 x - \delta_1 \sin\delta_2 x}{\delta_1\delta_2\left(\delta_2^2 - \delta_1^2\right)EI} \\ a_{14} = \dfrac{\cos\delta_1 x - \cos\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} \end{cases} \tag{41}$$

For $\omega_1(x)$, the following expressions are obtained, (42):

$$\begin{aligned} \omega_1(x) = & \left(-\frac{\delta_2^2\delta_1 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI} y_0^{'} - \frac{\delta_1 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI} M_0^{'}\right) + \left(\frac{\delta_2^2\delta_1 \cos\delta_1 x}{\delta_1\left(\delta_2^2 - \delta_1^2\right)EI}\omega_0^{'} + \frac{\delta_1 \cos\delta_1 x}{\delta_1\left(\delta_2^2 - \delta_1^2\right)EI} T_0^{'}\right) + \\ & + \left(\frac{\delta_1^2\delta_2 \sin\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} y_0^{'} + \frac{\delta_2 \sin\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} M_0^{'}\right) + \left(-\frac{\delta_1^2\delta_2 \cos\delta_2 x}{\delta_2\left(\delta_2^2 - \delta_1^2\right)EI}\omega_0^{'} - \frac{\delta_2 \cos\delta_2 x}{\delta_2\left(\delta_2^2 - \delta_1^2\right)EI} T_0^{'}\right) \end{aligned} \tag{42}$$

or (43):

$$\begin{aligned} \omega_1(x) = & \left(\frac{\delta_1^2\delta_2 \sin\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} - \frac{\delta_2^2\delta_1 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI}\right) y_0^{'} + \left(\frac{\delta_2^2\delta_1 \cos\delta_1 x}{\delta_1\left(\delta_2^2 - \delta_1^2\right)EI} - \frac{\delta_1^2\delta_2 \cos\delta_2 x}{\delta_2\left(\delta_2^2 - \delta_1^2\right)EI}\right)\omega_0^{'} + \\ & + \left(\frac{\delta_1 \cos\delta_1 x}{\delta_1\left(\delta_2^2 - \delta_1^2\right)EI} - \frac{\delta_2 \cos\delta_2 x}{\delta_2\left(\delta_2^2 - \delta_1^2\right)EI}\right) T_0^{'} + \left(\frac{\delta_2 \sin\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} - \frac{\delta_1 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI}\right) M_0^{'} \end{aligned} \tag{43}$$

and (44):

$$\begin{aligned} \omega_1(x) = & \frac{\delta_1\delta_2\left(\delta_1 \sin\delta_2 x - \delta_2 \sin\delta_1 x\right)}{\left(\delta_2^2 - \delta_1^2\right)EI} y_0^{'} + \frac{\delta_2^3 \cos\delta_1 x - \delta_1^3 \cos\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI}\omega_0^{'} + \\ & + \frac{\cos\delta_1 x - \cos\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} T_0^{'} + \frac{\delta_2 \sin\delta_2 x - \delta_1 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI} M_0^{'} \end{aligned} \tag{44}$$

Simplified, it can be written (45):

$$\omega_1(x) = a_{21} y_0^{'} + a_{22}\omega_0^{'} + a_{23} T_0^{'} + a_{44} M_0^{'} \tag{45}$$

with notations (46):

$$\begin{cases} a_{21} = \dfrac{\delta_1\delta_2\left(\delta_1 \sin\delta_2 x - \delta_2 \sin\delta_1 x\right)}{\left(\delta_2^2 - \delta_1^2\right)EI} \\ a_{22} = \dfrac{\delta_2^3 \cos\delta_1 x - \delta_1^3 \cos\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} \\ a_{23} = \dfrac{\cos\delta_1 x - \cos\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} \\ a_{24} = \dfrac{\delta_2 \sin\delta_2 x - \delta_1 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI} \end{cases} \tag{46}$$

For $T_1(x)$, we have (47):

$$\begin{aligned} T_1(x) = & \left(\frac{\delta_2^2\delta_1^3 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI} y_0^{'} + \frac{\delta_1^3 \sin\delta_1 x}{\left(\delta_2^2 - \delta_1^2\right)EI} M_0^{'}\right) + \left(-\frac{\delta_2^2\delta_1^3 \cos\delta_1 x}{\delta_1\left(\delta_2^2 - \delta_1^2\right)EI}\omega_0^{'} - \frac{\delta_1^3 \cos\delta_1 x}{\delta_1\left(\delta_2^2 - \delta_1^2\right)EI} T_0^{'}\right) + \\ & + \left(-\frac{\delta_1^2\delta_2^3 \sin\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} y_0^{'} - \frac{\delta_2^3 \sin\delta_2 x}{\left(\delta_2^2 - \delta_1^2\right)EI} M_0^{'}\right) + \left(\frac{\delta_1^2\delta_2^3 \cos\delta_2 x}{\delta_2\left(\delta_2^2 - \delta_1^2\right)EI}\omega_0^{'} + \frac{\delta_2^3 \cos\delta_2 x}{\delta_2\left(\delta_2^2 - \delta_1^2\right)EI} T_0^{'}\right) \end{aligned} \tag{47}$$

or (48):

$$T_1(x)=\left(\frac{\delta_2^2\delta_1^3\sin\delta_1 x}{(\delta_2^2-\delta_1^2)EI}-\frac{\delta_1^2\delta_2^3\sin\delta_2 x}{(\delta_2^2-\delta_1^2)EI}\right)y_0^{'}+\left(\frac{\delta_1^2\delta_2^3\cos\delta_2 x}{\delta_2(\delta_2^2-\delta_1^2)EI}-\frac{\delta_2^2\delta_1^3\cos\delta_1 x}{\delta_1(\delta_2^2-\delta_1^2)EI}\right)\omega_0^{'}+$$
$$+\left(\frac{\delta_1^3\sin\delta_1 x}{(\delta_2^2-\delta_1^2)EI}-\frac{\delta_2^3\sin\delta_2 x}{(\delta_2^2-\delta_1^2)EI}\right)M_0^{'}+\left(\frac{\delta_2^3\cos\delta_2 x}{\delta_2(\delta_2^2-\delta_1^2)EI}-\frac{\delta_1^3\cos\delta_1 x}{\delta_1(\delta_2^2-\delta_1^2)EI}\right)T_0^{'} \tag{48}$$

and (49):

$$T_1(x)=\frac{\delta_2^2\delta_1^2(\delta_1\sin\delta_1 x-\delta_2\sin\delta_2 x)}{(\delta_2^2-\delta_1^2)EI}y_0^{'}+\frac{\delta_1^2\delta_2^2(\cos\delta_2 x-\cos\delta_1 x)}{(\delta_2^2-\delta_1^2)EI}\omega_0^{'}+$$
$$+\frac{\delta_2^2\cos\delta_2 x-\delta_1^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI}T_0^{'}+\frac{\delta_1^3\sin\delta_1 x-\delta_2^3\sin\delta_2 x}{(\delta_2^2-\delta_1^2)EI}M_0^{'} \tag{49}$$

Simplified, it can be written (50):

$$T_1(x)=a_{31}y_0^{'}+a_{32}\omega_0^{'}+a_{33}T_0^{'}+a_{34}M_0^{'} \tag{50}$$

with notations (51):

$$\begin{cases} a_{31}=\dfrac{\delta_2^2\delta_1^2(\delta_1\sin\delta_1 x-\delta_2\sin\delta_2 x)}{(\delta_2^2-\delta_1^2)EI} \\ a_{32}=\dfrac{\delta_1^2\delta_2^2(\cos\delta_2 x-\cos\delta_1 x)}{(\delta_2^2-\delta_1^2)EI} \\ a_{33}=\dfrac{\delta_2^2\cos\delta_2 x-\delta_1^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI} \\ a_{34}=\dfrac{\delta_1^3\sin\delta_1 x-\delta_2^3\sin\delta_2 x}{(\delta_2^2-\delta_1^2)EI} \end{cases} \tag{51}$$

For $M_1(x)$, we have (44):

$$M_1(x)=\left(-\frac{\delta_2^2\delta_1^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI}y_0^{'}-\frac{\delta_1^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI}M_0^{'}\right)+\left(-\frac{\delta_2^2\delta_1^2\sin\delta_1 x}{\delta_1(\delta_2^2-\delta_1^2)EI}\omega_0^{'}-\frac{\delta_1^2\sin\delta_1 x}{\delta_1(\delta_2^2-\delta_1^2)EI}T_0^{'}\right)+$$
$$+\left(\frac{\delta_1^2\delta_2^2\cos\delta_2 x}{(\delta_2^2-\delta_1^2)EI}y_0^{'}+\frac{\delta_2^2\cos\delta_2 x}{(\delta_2^2-\delta_1^2)EI}M_0^{'}\right)+\left(\frac{\delta_1^2\delta_2^2\sin\delta_2 x}{\delta_2(\delta_2^2-\delta_1^2)EI}\omega_0^{'}+\frac{\delta_2^2\sin\delta_2 x}{\delta_2(\delta_2^2-\delta_1^2)EI}T_0^{'}\right) \tag{52}$$

or (53):

$$M_1(x)=\frac{\delta_1^2\delta_2^2(\cos\delta_2 x-\cos\delta_1 x)}{(\delta_2^2-\delta_1^2)EI}y_0^{'}+\frac{\delta_1\delta_2(\delta_1\sin\delta_2 x-\delta_2\sin\delta_1 x)}{(\delta_2^2-\delta_1^2)EI}\omega_0^{'}+$$
$$+\frac{\delta_2\sin\delta_2 x-\delta_1\sin\delta_1 x}{(\delta_2^2-\delta_1^2)EI}T_0^{'}+\frac{\delta_2^2\cos\delta_2 x-\delta_1^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI}M_0^{'} \tag{53}$$

Simplified, it can be written (54):

$$M_1(x)=a_{41}y_0^{'}+a_{42}\omega_0^{'}+a_{43}T_0^{'}+a_{44}M_0^{'} \tag{54}$$

with notations (55):

$$\begin{cases} a_{41}=\dfrac{\delta_1^2\delta_2^2(\cos\delta_2 x-\cos\delta_1 x)}{(\delta_2^2-\delta_1^2)EI} \\ a_{42}=\dfrac{\delta_1\delta_2(\delta_1\sin\delta_2 x-\delta_2\sin\delta_1 x)}{(\delta_2^2-\delta_1^2)EI} \\ a_{43}=\dfrac{\delta_2\sin\delta_2 x-\delta_1\sin\delta_1 x}{(\delta_2^2-\delta_1^2)EI} \\ a_{44}=\dfrac{\delta_2^2\cos\delta_2 x-\delta_1^2\cos\delta_1 x}{(\delta_2^2-\delta_1^2)EI} \end{cases} \tag{55}$$

The State Vector Associated with a Section *x* of the Bar on Elastic Environment

The state vector for section *x* of the bar on elastic environment can be written as a column vector with 4 elements, as follows (56):

$$\{V_1\}_x = \{V_1(x)\} = \{y_1(x), \omega_1(x), T_1(x), M_1(x)\}^{-1} \tag{56}$$

where:

- $\{V_1(x)\}_x$ is the state vector for section x
- $y_1(x)$ is the horizontal linear displacement of the section x
- $\omega_1(x)$ is the angular deformation near the x section
- $T_1(x)$ is the cutting force in section x
- $M_1(x)$ is the bending moment in section x.

We considered an orthogonal biaxial reference system with the origin at point *1*, with the vertical axis *x*, along symmetry axis of bar and with the horizontal axis *y*. The state vector, with index *0*, is associated in the origin, the section *0*, for $x=0$, (57):

$$\{V_1\}_0 = \{y'_0, \omega'_0, T'_0, M'_0\}^{-1} = \{V_1(0)\} = \{y'(0), \omega'(0), T'(0), M'(0)\}^{-1} \tag{57}$$

At the other end of bar for $x=l$, is associated the state vector (58):

$$\{V_1\}_l = \{V_1(l)\} = \{y'(l), \omega'(l), T'(l), M'(l)\}^{-1} \tag{58}$$

The Transfer-Matrix for a Buckling Bar on Elastic Environment as a Dental Implant

We can write the connection relation (59) between the origin section *0* and the section *x*, with the Transfer-Matrix $[Tm']_x$:

$$\{V_1\}_x = [Tm']_x \{V_1\}_0 + [Fe']_x \{Ve'\}_x \tag{59}$$

where:

- $\{V_1\}_x$ is the state vector corresponding to the section x
- $[Fe']_x$ is the matrix corresponding to the external loads in section x
- $\{Ve'\}_x$ is the state vector corresponding to the external loads in section x.

We can write the matrix relationship (59) and it is obtained (60):

$$\{V_1(x)\} = \begin{Bmatrix} y_1(x) \\ \omega_1(x) \\ T_1(x) \\ M_1(x) \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} y_0' \\ \omega_0' \\ T_0' \\ M_0' \end{Bmatrix} \tag{60}$$

For the section from the bottom, for $x=l$, relation (60) becomes (61):

$$\{V_1(l)\} = \begin{Bmatrix} y_1(l) \\ \omega_1(l) \\ T_1(l) \\ M_1(l) \end{Bmatrix} = \begin{bmatrix} a_{11}(l) & a_{12}(l) & a_{13}(l) & a_{14}(l) \\ a_{21}(l) & a_{22}(l) & a_{23}(l) & a_{24}(l) \\ a_{31}(l) & a_{32}(l) & a_{33}(l) & a_{34}(l) \\ a_{41}(l) & a_{42}(l) & a_{43}(l) & a_{44}(l) \end{bmatrix} \begin{Bmatrix} y_0' \\ \omega_0' \\ T_0' \\ M_0' \end{Bmatrix} \tag{61}$$

The conditions for the two extremes are the same (62):

$$\begin{cases} y_o' = 0 \\ M_0' = 0 \\ y_1(l) = 0 \\ \omega_1(l) = 0 \end{cases} \tag{62}$$

With (62), relation (61) becomes (63):

$$\begin{Bmatrix} 0 \\ 0 \\ T_1(l) \\ M_1(l) \end{Bmatrix} = \begin{bmatrix} a_{11}(l) & a_{12}(l) & a_{13}(l) & a_{14}(l) \\ a_{21}(l) & a_{22}(l) & a_{23}(l) & a_{24}(l) \\ a_{31}(l) & a_{32}(l) & a_{33}(l) & a_{34}(l) \\ a_{41}(l) & a_{42}(l) & a_{43}(l) & a_{44}(l) \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_0' \\ T_0' \\ 0 \end{Bmatrix} \tag{63}$$

We have now a linear system of 4 equations with 4 unknowns (64):

$$\begin{cases} a_{12}(l)\cdot\omega_0^{'}+a_{13}(l)\cdot T_0^{'}=0 \\ a_{22}(l)\cdot\omega_0^{'}+a_{23}(l)\cdot T_0^{'}=0 \\ a_{32}(l)\cdot\omega_0^{'}+a_{33}(l)\cdot T_0^{'}=T_1(l) \\ a_{42}(l)\cdot\omega_0^{'}+a_{43}(l)\cdot T_0^{'}=M_1(l) \end{cases} \tag{64}$$

Solutions of system (64) will allow us to determine the value of critical buckling force in case of a bar embedded at one end on an elastic environment and with a guided joint at the other end, subjected to compression by a vertical axial force, with which dental implant is assimilated into bone, bone being likened to an elastic environment. TMM is a very advantageous method, because it offers the possibility to program the presented calculation algorithm and thus, offers quick solutions in exceptional situations in orthodontics.

Results and Discussion

The results obtained after solving the system (64) gives us the opportunity and will allow us to determine all the components of the state vector of the section at the origin, the elements from $x=0$ and of the state vector corresponding to the section $x=l$. Finally, it can be determined, like this, the value of critical buckling force in case of a bar embedded at one end on an elastic environment and with a guided joint at the other end, subjected to compression by a vertical axial force, with which dental implant is assimilated into bone, bone being likened to an elastic environment. Due to rapid evolution in computerization of all medicine branches, applications in orthodontics of mathematical approach for dental implants are in full development. For our future research, this approach is fundamental for studies about titanium and its alloys of medicine domains, for prosthesis, in occurrence, in orthodontics, for dental implants. Approach for this problem of buckling bars on elastic environment is analyzed by TMM, providing a validation basis for future experimental tests of some of bio-composite materials. After, we want to validate the algorithm by Finite Element Method (FEM). Due to mechanical properties like the bone tissue, titanium and its alloys are most widely used: a good hardness and a good corrosion resistance. For straight bars compressed with an axial force, risks of buckling are higher, which makes very important knowing the critical buckling force. We considered the dental implant as a bar on elastic environment, one-end of bar embedded and other end as a guided joint, with an axial compression force P at the joint end. The bone is assimilated as an elastic environment, that is an original idea and approaching and solving this problem with TMM is also an original idea. At the beginning, a calculation was presented an analytical calculus for buckling straight bars with axial compression force on rigid environment with the average deformed fiber for a bar, which rests on a rigid environment**.** It was then defined the state vector associated with a section x of the bar which rests on a rigid environment and the Transfer-Matrix associated of a straight bar. After, the algorithm calculus was given for a dental implant as a buckling straight bar with axial compression force on elastic environment and the average deformed fiber for a bar, which rests on elastic environment was given too. It was then defined a state vector associated with a section x of the bar on elastic environment and the Transfer-Matrix for buckling bars on elastic environment as a dental implant. TMM is a very advantageous method, because it offers the possibility to program the presented calculation algorithm and thus, offers quick solutions in exceptional situations in orthodontics.

Conclusion

In this paper, we present an original relatively simple analytical calculus for critical buckling force. That is an original idea to assimilate the dental implant as a bar embedded in an elastic environment. The algorithm is based on the mathematical approach offered by

Dirac's and Heaviside's functions and operators, with directly applications for dental implants. Studies about dental implants are very important for orthodontic. The concern of researchers for the development and improvement of dental implants is very broad, as can be seen from the numerous publications in the field, some of which have been cited in this work.

This work presents a calculus method with an approach by TMM of buckling bar with which the dental implant was assimilated. Bone was considered as an elastic environment. That is an original idea to assimilate the dental implant as a bar embedded in an elastic environment. TMM is very easy to program and so we can obtain instant results with quick application in practice, even in situ. This work with this approach will serve for o future research about dental implants, for optimization forms of implants. In the future, we want to validate the results obtained through TMM with other theoretical and experimental methods.

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